MATH 25000: Calculus III Lecture Notes

Dr. Amanda Harsy

October 17, 2017
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35.1 MATH 25000 – HW 1 Due Friday, 9/1

35.2 MATH 25000 – HW 2 Due Friday, 9/15

35.3 MATH 25000 – HW 3 Due Tuesday, 9/26

35.4 MATH 25000 – HW 4 Due Friday, 10/13

35.5 MATH 25000 – HW 5 Due Friday, 10/27

35.6 MATH 25000 – HW 6 Due Friday, 11/10

35.7 MATH 25000 – HW 7 due Monday: 11/20

35.8 MATH 25000 – HW 8 due Monday: 12/4

35.9 MATH 25000 – HW 9: -Due Tuesday 12/5
1 Syllabus and Schedule

Thanks for taking Calculus III with me! It is one of my favorite classes to teach and I think it is a great way to end your Calculus Sequence. Now, you may be asking yourself (or have asked yourself), “What is Calculus, and why do I have to take this class?” Calculus is, in my opinion, ultimately is the study of change. In particular, calculus gives us the tools to be able to understand how changing one or more linked variables reflects change in other variables[1]. In other words, Calculus is the study and modeling of dynamical systems[2]. In Calculus I, we learned about the derivative of a function and some of its applications. Recall, a derivative is a measure of sensitivity of change in one variable to change in the other - the instantaneous rate of change. When we learn about integration, we are measuring accumulation or the limit of a summation of smaller parts[3].

In Calculus II, we built upon this idea that we can use integrals to calculate and model complex situations by accumulating the sums of simpler parts. We also learned techniques used in calculating and approximating these integrals and discuss ways of modeling functions and infinite systems.

Calculus III should really be renamed, *The Greatest Hits of Calculus*. We revisit all of the amazing theory we learned in Calculus I and II, but now we just generalize it to the multivariate setting. We also generalize it to Vector Fields at the end of the course. At times during this course, the topics may seem disjointed. For example, we start the semester with parametric equations and an introduction to vectors. Differentiation and integration is still there, but isn’t the main event during this time. We then get into the greatest hits part of Calc 3 and revisit differentiation and integration. At this point in the course, you may think, wait, but what about the vectors? Don’t worry. Our last month will be combining the multivariate calculus with vector calculus and this culminates in several important theorems which tie all of Calculus III topics together into several beautiful and useful packages!

I hope you will enjoy this semester and learn a lot! Please make use of my office hours and plan to work hard in this class. My classes have a high work load (as all math classes usually do!), so make sure you stay on top of your assignments and get help early. Remember you can also email me questions if you can’t make my office hours or make an appointment outside of office hours for help. When I am at Lewis, I usually keep the door open and feel free to pop in at any time. If I have something especially pressing, I may ask you to come back at a different time, but in general, I am usually available. The HW Assignments, and Practice Problems for Exams are at the end of this course packet. I have worked hard to create this course packet for you, but it is still a work in progress. Please be understanding of the typos I have not caught, and politely bring them to my attention so I can fix them for the next time I teach this course. I look forward to meeting you and guiding you through the magnificent course that is Calculus III.

Cheers,
Dr. H

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Acknowledgments: No math teacher is who she is without a little help. I would like to thank my own undergraduate professors from Taylor University: Dr. Ken Constantine, Dr. Matt Delong, and Dr. Jeremy Case for their wonderful example and ideas for structuring excellent learning environments. I also want to thank the members from both the MathVote Projects for sharing some of their clicker questions. And finally, I would like to thank you and all the other student for making this job worthwhile and for all the suggestions and encouragement you have given me over the years to improve.
Instructor: Amanda Harsy
Email: harsyram@lewisu.edu
Office: AS-120-A
Office phone: 815-836-5688
Class Time Section 1: MTWF 1-1:50
Class Room Section 1: AS-157-A
Class Time Section 2: MTWF 2-2:50
Class Room Section 2: AS-157-A
Office Hours: M 10-11, W: 10-11, R 12-2, F:3-4 or by appt

Course description

This is a 4 credit course which provides a study of Euclidean vector spaces, conic sections, other coordinate systems, parameterized curves and functions of several variables. Differential and integral calculus for functions involving vectors, along with their applications, is presented.

About the Course


Prerequisites: A grade of C or better in Calculus II (13-201).

Course Objectives: To help each student

- understand and use concepts of multivariate calculus,
- apply multivariate calculus concepts to various applications,
- become better at solving mathematical problems, experience doing mathematics co-operatively,
- effectively use technology to study mathematics,
- effectively communicate mathematics, and
- enjoy his/her experience with mathematics.

Resources

Blackboard: Check the Blackboard website regularly (at least twice a week) during the semester for important information, announcements, and resources. It is also where you will find the course discussion board. Also, check your Lewis email account every day. I will use email as my primary method of communication outside of office hours.

Help: Don’t wait to get help. Come to my office hours! Go to the Math Study Tables! Go to LARC! Get a tutor! Find a study buddy! My job is to help you, so please don’t hesitate to email me, post on the discussion board, or stop by my office or schedule an appointment if my office hours don’t work for you.
Dr. Harsy’s web page: On many of the topics we will cover this year, I have echo pen lectures you can access and listen to at any time. This may help if you miss a class. You can access these “playable pdf’s” at http://www.cs.lewisu.edu/~harsyram/echopen.html. Let me know if you have issues playing them.

**Student Learning Outcomes**

Upon successful completion of this course, students should be able to

- Perform standard operations on vectors in two-dimensional space and three-dimensional space.
- Compute the dot product of vectors, lengths of vectors, and angles between vectors.
- Compute the cross product of vectors and interpret it geometrically.
- Determine equations of lines and planes using vectors.
- Identify various quadric surfaces through their equations.
- Sketch various types of surfaces by hand and by using technology.
- Define vector functions of one real variable and sketch space curves.
- Compute derivatives and integrals of vector functions.
- Find the arc length and curvature of space curves.
- Find the velocity and acceleration of a particle moving along a space curve.
- Define functions of several variables and their limits.
- Calculate partial derivatives of functions of several variables.
- Interpret partial derivatives graphically.
- Apply the chain rule for functions of several variables.
- Calculate the gradient and directional derivatives of functions of several variables.
- Solve problems involving tangent planes and normal lines.
- Determine and classify the extrema of functions of several variables.
- Use the Lagrange multiplier method to find extrema of functions with constraints.
- Define double integrals over rectangles.
- Compute iterated integrals.
- Define and compute double integrals over general regions.
- Compute double integrals in polar coordinates.
- Compute triple integrals in Cartesian coordinates, cylindrical coordinates, and spherical coordinates.
- Apply triple integrals to find volumes.
- Sketch and interpret vector fields.
- Calculate line integrals along piecewise smooth paths, and interpret such quantities as work done by a force.
- Use the fundamental theorem of line integrals.
- Compute the curl and the divergence of vector fields.
- Determine whether a vector field is conservative.
- Compute surface integrals.
- Use double, triple and line integrals in applications, including Green’s Theorem, Stokes’ Theorem and the Divergence Theorem.

**Additional topics:** These are at the instructor’s discretion as time allows. Some suggested additional topics are finding moments and centers of mass, change of variables in multiple integrals, and translation of axes.
Course Requirements And Grading Policy

**Turn In Homework:** Almost every other week, I will collect a homework assignment which will consist of 3-5 problems. I will post these homework assignments on Blackboard. You can think of these assignments as “take-home” quizzes. You may work with others on the homework, but it must be your own work. If I catch you copying homework, you will get a 0. Please see the academic honesty section below.

**Daily WebAssign Homework:** The daily Webassign homework will count towards your homework grade. Make sure you have a book with an access code. I will use problems similar to the Webassign homework for your quizzes. You can also do book problems for extra practice since they usually have solutions in the back of the text. Most WebAssign assignments are due Tuesdays and Fridays.

**Quizzes:** There will quizzes periodically over the WebAssign homework. These will cover any section learned before the quiz day. Quizzes with solutions will be published on Blackboard after the quiz date. There will be no make up Quizzes due to the answers being published. I may add a pop (unscheduled) quiz.

**Gateway:** Since you will be able to use calculators on almost every test and quiz, we need you to demonstrate that you do have the integration ability to be able to solve integral by yourself. Thus there will be a short, no-calculator Gateway Exam on Integral Proficiency. Each student will be required to obtain a minimum score on the exam (a 9/10). The exam can be repeated as needed, but cannot be taken more than once a day. You can take the gateway during my office hours, during the Math Study Tables, or schedule a time with me through email. Each student passing the Gateway Exam by Exam 1 will receive a 2% increase in their homework grade. Those who pass the Gateway after Exam 1 and by Exam 2 will receive 1% increase in their homework grade. You will need to complete the gateway by **October 31st.** The Gateway Exam is worth 7% of your grade and is graded as a Pass/Fail Exam.

**Discussion Board Participation:** Mathematics is not learned in isolation. You must communicate with others about its use how to solve problems. Work together. One way we will all work together in this class is by using a “Discussion Board in Blackboard. Whenever you have a question about a problem, or if something in the textbook is not quite clear, you should post the question into an appropriate discussion forum. This allows everybody to get familiar with the other members of the class, and it allows me to answer a single question once and for all. You will get a small amount of extra credit towards participation. You can get points several difference ways. One way is to ask a question from the homework or class and explaining what steps you took towards solving the problem and where you got stuck. Another way is to help answer another student’s question or a question that I have posted. I may respond by attaching an echo pen pdf. You can also write your question and take a picture of it or scan it and post it on the forum. Blackboard has a /nice editor you can use by clicking the down arrows on the right side of the text box. This will then show more options for writing in the forum. You can then click the $f_x$ symbol to pull up the math editor.
Mastery-Based Testing: This course will use a testing method called, “Mastery-Based Testing.” There will be four (4) paper-and-pencil, in-class Mastery Exams given periodically throughout the semester. In mastery-based testing, students receive credit only when they display ”mastery”, but they receive multiple attempts to do so. The primary source of extra attempts comes from the fact that test questions appear on every subsequent test. In this Calculus III course, Test 1 will have 5 questions. Test 2 will have 10 questions -a remixed version of the five from Test 1 and five new questions. Test 3 will have 15 questions- a remixed version of the ten from Test 2 and five new questions. Test 4 will have 18 questions- a remixed version of the fifteen from Test 3 and three new questions. We will also have four testing weeks. During these weeks, students can use doodle to sign up to retest concepts during (extended) office hours. Students are allowed to test any concept, but cannot retest that concept the rest of the week. So for example, a student can test concepts 2,3 and 5 on Monday and concept 6 on Tuesday, but would not be able to test concept 5 again. These testing weeks are tentatively set for 10/9-10/13, 11/6-11/10, 11/27-12/1, and during Finals week.

Grading of Mastery-Based Tests: The objectives of this course can be broken down into 18 main concepts/problems. For each sort of problem on the exam, I identify three levels of performance: master level, journeyman level, and apprentice level. I will record how well the student does on each problem (an M for master level, a J for journeyman level, a 0 for apprentice level) on each exam. After the Final testing week, I will make a record of the highest level of performance the student has made on each sort of problem or project and use this record to determine the student’s total exam grade. Each of the first 9 concept/questions students master will count 8% points towards their exam grade (with the first one being worth 9%). After that, each concept/question will be worth 3% towards your exam grade. So for example, if you master 11 of the 18 concepts your grade will be a 79%.

This particular way of arriving at the course grade is unusual. It has some advantages. Each of you will get several chances to display mastery of almost all the problems. Once you have displayed mastery of a problem there is no need to do problems like it on later exams. So it can certainly happen that if you do well on the midterms you might only have to do one or two problems on the Final. (A few students may not even have to take the final.) On the other hand, because earlier weak performances are not averaged in, students who come into the Final on shaky ground can still manage to get a respectable grade for the course. This method of grading also stresses working out the problems in a completely correct way, since accumulating a lot of journeyman level performances only results in a journeyman level performance. So it pays to do one problem carefully and correctly as opposed to trying get four problems partially correctly. Finally, this method of grading allows you to see easily which parts of the course you are doing well with, and which parts deserve more attention. The primary disadvantage of this grading scheme is that it is complicated. At any time, if you are uncertain about how you are doing in the class I would be more than glad to clarify the matter. The tentative test dates are below.
Exam # | Date
---|---
1 | 9/29
2 | 10/25
3 | 11/21
4 | 12/6

**Grade scale**

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<tr>
<td>90%-93%</td>
<td>A-</td>
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<tr>
<td>93%-96%</td>
<td>A</td>
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<tr>
<td>&gt; 97%</td>
<td>A+</td>
</tr>
<tr>
<td>80%-83%</td>
<td>B-</td>
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<tr>
<td>83%-86%</td>
<td>B</td>
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<tr>
<td>87%-89%</td>
<td>B+</td>
</tr>
<tr>
<td>70%-72%</td>
<td>C-</td>
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<tr>
<td>73%-76%</td>
<td>C</td>
</tr>
<tr>
<td>77%-79%</td>
<td>C+</td>
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<tr>
<td>67%-69%</td>
<td>D+</td>
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**Course Policies and Procedures**

**Calculator Policy:** You will find it useful to have a graphing calculator. I recommend a TI-Nspire CX CAS or a TI-89, but do not buy a new calculator if you already have a close equivalent. The graphing calculators will be allowed on *almost every* test and quiz. No other forms of technology can be used on in-class, closed-books assessments (quizzes, tests, final). The Math Study Tables will have a few TI-Nspires available during the Study Table hours.

**Phones:** Please do not have your phones out *unless it is a class activity*. In class, we may sometimes use Poll Everywhere and during this time you can have your phone out. If I see a phone out during a quiz, you will receive an F on that quiz.

**Make-Ups:** There will be no make-ups for any assignments. If you are late or miss class, your assignment will not be accepted and there will be no make-up offered, except in extenuating and unpredictable circumstances. If you will miss class for a justifiable & unavoidable reason, you can contact me **before** you miss class & it is possible you can have a make-up. If you do not contact me & explain your absence, you will not be allowed a make-up.

**Class Attendance:** Students are expected to attend all classes as part of the normal learning process. In addition, students must be especially consistent in attendance, both on-ground and online, during the first two weeks of the semester to confirm registration and to be listed on the official course roster. Students who fail to follow this procedure and who have not received prior approval from the instructor for absences will be withdrawn from the courses in question by certification of the instructor on the official class lists. Instructors may publish specific, additional reasonable standards of attendance for their classes in the course syllabus. Students may receive failing grades if they do not observe attendance requirements. The Illinois Student Assistance Commission also requires attendance as a demonstration of academic progress toward a degree as one criterion for retaining financial aid awards. (2015-2016 Undergraduate Catalog, p. 34).
Attendance is critical for success in this course and is required of all students without exception. A student absent from class is responsible for all material covered that day. I usually post solutions to ICE sheets and post notes. I will stop doing this if there is a trend of absences. In other words, I will not post lecture notes unless there is consistent attendance. Students are expected to attend all classes as part of the normal learning process.

**Course Relationship to Mission:** Lewis University is a Catholic University in the Lasallian Tradition. Our Mission is integrated into all aspects of University life, including this course. This course embraces the Mission of the University by fostering an environment in which each student is respected as an individual within a community of learners. In the spirit of the vision of Lewis University, the goals and objectives of this course seek to prepare students to be successful, life-long learners who are intellectually engaged, ethically grounded, socially responsible, and globally aware.

**Academic Honesty:** Scholastic integrity lies at the heart of Lewis University. Plagiarism, collusion and other forms of cheating or scholastic dishonesty are incompatible with the principles of the University. Students engaging in such activities are subject to loss of credit and expulsion from the University. Cases involving academic dishonesty are initially considered and determined at the instructor level. If the student is not satisfied with the instructors explanation, the student may appeal at the department/program level. Appeal of the department /program decision must be made to the Dean of the college/school. The Dean reviews the appeal and makes the final decision in all cases except those in which suspension or expulsion is recommended, and in these cases the Provost makes the final decision.

**Classroom Decorum:** In order to maintain an environment conducive to learning and student development, it is expected that classroom discourse is respectful and non-disruptive. The primary responsibility for managing the classroom environment rests with the faculty. Students who engage in any prohibited or unlawful acts that result in disruption of a class may be directed by the faculty member to leave class for the remainder of the class period. Students considered to be a disruption or who present a threat of potential harm to self or others may be referred for action to the Dean of Student Services. 2015-2016 Student Handbook, p. 14, Lewis University website https://www.lewisu.edu/sdl/pdf/studenthandbook.pdf.

**Sanctified Zone:** This learning space is an extension of Lewis Universityys Sanctified Zone, a place where people are committed to working to end racism, bias and prejudice by valuing diversity in a safe and nurturing environment. This active promotion of diversity and the opposition to all forms of prejudice and bias are a powerful and healing expression of our desire to be Signum Fidei, Signs of Faith, in accordance with the Lewis Mission Statement. To learn more about the Sanctified Zone, please visit: http://www.lewisu.edu/sanctifiedzone.

**Students Requiring Special Accommodations:** Lewis University is committed to providing equal access and opportunity for participation in all programs, services and activities. If you are a student with a disability who would like to request a reasonable accommodation, please speak with the Learning Access Coordinator at the Center for Academic Success and Enrichment. Please make an appointment by calling 815-836-5593 or emailing learningaccess@lewisu.edu. For more information about academic support services, visit the website at: www.lewisu.edu/CASE. Since accommodations require early planning and are not provided retroactively, it is recommended that you make your request prior to or during the first week of class. It is not necessary to disclose the nature of your disability to your instructor.
CLASS NOTATION

- $\mathbb{R}$ represents all real numbers (so no imaginary numbers)
- $\mathbb{Z}$ represents integer numbers
- $\mathbb{Q}$ represents rational numbers [numbers that can be written as a fraction $\frac{p}{q}$ where $p,q$ are integers]
- $\{ \}$ represents a set. For example $\{x; \text{such that } x \text{ is an integer }\} = \mathbb{Z}$
- $\in$ means “contained in”. For example, $5 \in \mathbb{R}$
- $\notin$ means “not contained in”. For example $\pi \notin \mathbb{Z}$.
- $\therefore$ means “such that”
- “|” or “:” mean “such that” in a set setting. For example $\{x|x \in \mathbb{Z}\} = \{x; \text{such that } x \text{ is an integer }\} = \mathbb{Z}$
- $\therefore$ means “therefore”.
- $\exists$ means “there exists”
- $\forall$ means “for all”
- $\Rightarrow$ means “this implies”
- $\iff$ or “iff” means “if and only if”
- Pf means “Proof”

This, That, Tips, and Suggestions...

Here are some suggestions that may help you in the class.

**Situation:** I am a student that learn best from writing everything down and I wish the lecture notes weren’t so typed out.

**Suggestion:** Writing is the best way for me to remember things too! I want to encourage you to write if it helps you. Using lecture notes keep me somewhat organized and helps us get through all of the material (of which there is a ton)! Also, my handwriting is horrible so having some words typed is good for you. My suggestion is to rewrite your lecture notes into a notebook. Rewriting notes is a wonderful study tool and I encourage you to do this if you don’t think the partially filled out notes allow you to learn best.

**Situation:** I can’t see the document camera.

**Suggestion:** I am used to higher quality document cameras and am waiting to get a better one. I have horrible board handwriting so I don’t like to write on the board for the majority of class. I do write on the board sometimes. Also I like the fact that the document camera allows me to face the class, so hopefully I can keep tabs on how the class is doing. My suggestion for you is to move to a different seat if you can’t see the board. And also please let me know if you are having issues and I can try to refocus/adjust the document camera.

**Practice, Practice, Practice:** The most common suggestion from past students is to do ALL the book homework. I pick quiz problems directly from these homework problems, so you should do the homework just for that reason. But you should also do them because learning mathematics is like any activity and requires that you practice in order to become proficient.
Topics Covered

- Conic Sections
- Parametric Curves
- Polar Coordinates and Equations
- Graphing Polar Equations
- Calculus with Polar Equations
- Cartesian Coordinate in 3-Space
- Vectors
- Dot Products
- Cross Products
- Vector Valued Functions
- Lines and Tangent Lines in 3-Space
- Curvature
- Components of Acceleration
- Surfaces (paraboloids, cones, etc)
- Cylindrical Coordinates
- Spherical Coordinates
- Contour Graphs and Level Curves
- Partial Derivatives
- Limits of multivariate functions
- Differentiability and continuity of multivariate functions
- Directional Derivatives
- Chain Rule For Multivariate Functions
- Tangent Plane and Differentials of multivariate functions
- Locating and Classifying Max and Mins
- Lagrange Multipliers
- Double Integrals
- Fubini’s Theorem
- Iterated Integrals
- Integrals with Polar Coordinates
- Center of Mass
- Surface Area
- Triple Integrals
- Triple integrals with Spherical and Cylindrical Coordinates
- Change of Variable (Jacobeian)
- Vector Fields
- Line Integrals
- Fundamental Theorem of Line Integrals
- Conservative Vector Fields
- Finding Potential Functions
- Curl & Divergence
- Recognizing gradient vector fields
- Green’s Theorem
- Surface Integrals
- Gauss’s Divergence Theorem
- Stokes’ Theorem

Additional topics: These are at the instructor’s discretion as time allows. Some suggested additional topics are finding moments and centers of mass, change of variables in multiple integrals, and translation of axes.
<table>
<thead>
<tr>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
<th>Friday</th>
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The table above outlines the tentative topics to be covered each day and quiz and homework due dates. You also have WebAssign HW due most Tuesdays and Thursdays. This schedule is subject to change.
Assessment and Mapping of Student Learning Objectives:  
**Baccalaureate Characteristics:**

BC 1. The baccalaureate graduate of Lewis University will read, write, speak, calculate, and use technology at a demonstrated level of proficiency.

**Measurable Student Learning Outcome:**
*Advocate for a cause or idea, presenting facts and arguments, in an organized and accurate manner using some form of technology. Include qualitative and quantitative reasoning.*

<table>
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<th>Course Student Learning Outcomes</th>
<th>Baccalaureate Characteristics</th>
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<td>1. Perform standard operations on vectors in two-dimensional space and three-dimensional space.</td>
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<tr>
<td>2. Compute the dot product of vectors, lengths of vectors, and angles between vectors.</td>
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<td>3. Compute the cross product of vectors and interpret it geometrically.</td>
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<td>4. Determine equations of lines and planes using vectors.</td>
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<td>5. Identify various quadric surfaces through their equations.</td>
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<td>6. Sketch various types of surfaces by hand and by using technology.</td>
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<td>7. Define vector functions of one real variable and sketch space curves.</td>
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<td>8. Compute derivatives and integrals of vector functions.</td>
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<td>9. Find the arc length and curvature of space curves.</td>
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<td>10. Find the velocity and acceleration of a particle moving along a space curve.</td>
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<td>11. Define functions of several variables and their limits.</td>
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<td>12. Calculate partial derivatives of functions of several variables.</td>
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<td>13. Interpret partial derivatives graphically.</td>
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<td>14. Apply the chain rule for functions of several variables.</td>
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<td>15. Calculate the gradient and directional derivatives of functions of several variables.</td>
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<td>16. Solve problems involving tangent planes and normal lines.</td>
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<td>17. Determine and classify the extrema of functions of several variables.</td>
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<td>18. Use the Lagrange multiplier method to find extrema of functions with constraints.</td>
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<td>19. Define double integrals over rectangles.</td>
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<td>20. Compute iterated integrals.</td>
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<td>21. Define and compute double integrals over general regions.</td>
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<td>22. Compute double integrals in polar coordinates.</td>
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<td>23. Compute triple integrals in Cartesian coordinates, cylindrical coordinates, and spherical coordinates.</td>
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<td>24. Apply triple integrals to find volumes.</td>
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<td>25. Sketch and interpret vector fields.</td>
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<td>26. Calculate line integrals along piecewise smooth paths, and interpret such quantities as work done by a force.</td>
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<td>27. Use the fundamental theorem of line integrals.</td>
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<td>28. Compute the curl and the divergence of vector fields.</td>
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<td>29. Determine whether a vector field is conservative.</td>
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<td>30. Compute surface integrals.</td>
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<td>31. Use double, triple and line integrals in applications, including Green's Theorem, Stokes' Theorem and the Divergence Theorem.</td>
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2 Sample Gateway

1. \[ \int x \ln(x^2 + 5) \, dx \]

2. \[ \int \frac{z^2 - 4}{z^3} \, dz \]

3. \[ \int \frac{e^x}{e^x - 2} \, dx \]

4. \[ \int \cos^2 x \, dx \]

5. \[ \int \sqrt{t - 3} \, dt \]
6. \[ \int \frac{dx}{\sqrt{1-x^2}} \]

7. \[ \int \frac{1}{(x+3)(x-1)} \, dx \]

8. \[ \int \frac{1}{y} (\ln y)^3 \, dx \]

9. \[ \int \sin(3\theta) \, d\theta \]

10. \[ \int e^x \sin x \, dx \]
3 Parametric Equations

Sometimes curves cannot be written as a function in one variable. For example, a circle fails the vertical line test. One way to deal with this is to write $x$ and $y$ in terms of another variable, usually $t$. This is called a parametric representation or equation. For example, we can represent the points of the unit circle using parametric equations. The standard way to do this is to represent each point $(x, y)$ on the circle by $\cdots$.

Example 1: Sketch the curve described by the equations $x = 2 - 3t$, $y = 1 + t$. Then find a Cartesian equation of the curve.

Voting Question 1: Find a parametric representation for the curve $y = x^2$.
(a) $x(t) = t$ and $y(t) = t^2$
(b) $x(t) = -t$ and $y(t) = t^2$
(c) $x(t) = 1 - t$ and $y(t) = 1 - 2t + t^2$
(d) more than one of the above
(e) all of the above
3.1 Calculus with Parametric Equations

Remember our beloved chain rule: The derivative of y with respect to t, \( \frac{dy}{dt} = \)

Thus \( \frac{dy}{dx} = \)

This formula allows us to find the derivative of a parametric curve without having to eliminate the parameter t.

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( \frac{dy}{dt} \cdot \frac{dt}{dx} \right) = \frac{d}{dt} \left( \frac{dy}{dx} \right) \frac{dx}{dt}
\]

Example 2: Find the equation of the tangent(s) to the curve \( x = 1 + t^2, \ y = t + \frac{1}{t} + 3 \) at \( t = 1 \).

Objective 3: Area

If we have a curve with parametric equations \( x = f(t) \) and \( y = g(t) \) for \( \alpha \leq t \leq \beta \)

the area under the curve is \( A = \int_{a}^{b} ydx = \) or \( \int_{a}^{b} xdy = \)

Example 3: Set up an integral that represents the area enclosed by the curve \( x = 1 + e^t, \ y = t - t^2 \) and the x-axis.
3.2 Arc Length

A curve with parametric equations \( x = f(t) \) and \( y = g(t) \) for \( \alpha \leq t \leq \beta \) where \( f' \) and \( g' \) are continuous and \( \frac{dy}{dx} = f'(t) > 0 \) (which means our curve travels once from left to right as \( t \) increases from \( \alpha \) to \( \beta \)) has an Arc Length=

**Example 4:** Set up an integral that represents the length of the curve \( x = t + \sqrt{t}, \ y = t - \sqrt{t} \) \( 0 \leq t \leq 1 \).

**Example 5:** Find a Cartesian equation of the curve \( x = \sqrt{1+t}, \ y = \sqrt{t} - 1 \), sketch the curve, and indicate with an arrow the direction in which the curve is traced as the parameter increases.
3.3 ICE 1 - Parametric Equations

1. a) Find the slope of the tangent line to \( x = \ln(t) \), \( y = \sqrt{t + 1} \) at \( t = 1 \).

b) Find the \( \frac{d^2y}{dx^2} \)

2. a) Sketch the curve described by \( x = 2 \sin t \), \( y = 2 \cos t \) (you can use a calculator).

b) Find the distance traveled by \( (x, y) \) as \( t \) varies over \( 0 \leq t \leq 3\pi \) for \( x = 2 \sin t \), \( y = 2 \cos t \). Compare it with the length of the curve.
4 3-Space

4.1 Cartesian Coordinates in 3 Space

Right Hand Rule:

Points will be in the form:

Labeling Axes is VERY Important!

The $xy$ plane: the ___ coordinate is always
The $xz$ plane: the ___ coordinate is always
The $yz$ plane: the ___ coordinate is always

Example 1: On the graph above, plot the points, the origin, $Q = (-5, -5, 7)$, and $P = (3, 0, 5)$.

Two points determine a rectangular box called a _______________. (See next page to see the parallelepiped created by $Q$ and the origin.)

Distance Formula:

Midpoint Formula:

Equation of a Sphere:

Treat spheres like ellipses, but now we don’t set one side equal to 1. Often we must _______ _______ _______ __________ to find the center.
Voting Question 1: You awaken one morning to find that you have been transferred onto a grid which is set up like a standard right-hand coordinate system. You are at the point \((-1,3,-3)\), standing upright, and facing the xz-plane. You walk 2 units forward, turn left, and walk for another 2 units. What is your final position?

(a) \((-1,1,-1)\)
(b) \((-3,1,-3)\)
(c) \((-3,5,-3)\)
(d) \((1,1,-3)\)

Voting Question 2: Which of the following points lies closest to the xy-plane?

(a) \((3,0,3)\)
(b) \((0,4,2)\)
(c) \((2,4,1)\)
(d) \((2,3,4)\)

4.2 Linear Equations and Traces

Equation of a Linear Equation is of the form:
where \(A^2 + B^2 + C^2 \neq 0\) (aka A,B, and C are not all 0)
Fact: The graph of a linear equation is a ________.


Definition: Traces are the line of intersection of a given plane with the coordinate axes.

Example 2: Sketch the graph of $6x - 3y + 15z = 3$

Example 3: Sketch the graph of $x = 4$

4.3 Parametric Equations in 3 Dimensions

We can easily extend parametric equations for 3 variables $x = f(t)$, $y = g(t)$, $z = h(t)$.

Definition: A curve is smooth if

Arc Length for 3-Space: For curve described by smooth curves, $x = f(t)$, $y = g(t)$, $z = h(t)$ is $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$
5 Vectors

5.1 Introduction to Vectors

What is the difference between a vector and a scalar?

Examples of Scalars: Speed, work, energy, mass, length
Examples of Vectors: Velocity, force, torque, displacement

We can denote vectors with ________

We can denote vectors using components:

And we can denote vectors using basis vectors

Definition: Our standard unit vectors or basis vectors are

\[ i = \text{__________}, \quad j = \text{__________}, \quad \text{and} \quad k = \text{__________} \]

Definition: The zero vector is denoted by ________ and is the only vector without a

We can add or subtract vectors:

Triangle Law Parallelogram Law Component-wise

Warning: \( < -3, 0, 5 > \) is different from \( (-3, 0, 5)! \)

And we can multiply a vector by a ________________.

This is called “______________ multiplication”
**Warning:** $<1,2,5> \cdot <2,7,-1> \neq$

**Example 1:** a) How does multiplying $u$ by the scalar 2 change $u$?

b) How does multiplying $u$ by the scalar -1 change $u$?

c) What is $3u$ if $u = -i + j + 2k$?

d) Add $3u - v$ if $u = < -1, 1, 2 >$, and $v = < -3, 0, 5 >$. Write the solution in terms of our basis vectors.

**Definition:** The *length* or magnitude of a vector $u$ is denoted $\|u\|$ or $|u|$. The magnitude of vector $u = < u_1, u_2, u_3 >$ is calculated by determining the distance between the point at the head of the vector, $(u_1, u_2, u_3)$ and the origin. Thus $|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$.

**Definition** A vector having length one is called a *unit* vector.

**Properties of Vectors**

Most of our basic rules apply to vectors.

1. $u + v = v + u$
2. $u + 0 = 0 + u$
3. $a( bu ) = (ab) u$
4. $(a + b)u = au + bu$
5. $|a| |u| = |au|$
6. $(u + v) + w = u + (v + w)$
7. $u + (-u) = 0$
8. $a( u + v) = au + av$
9. $1u = u$
**Property:** Two vectors are parallel if they point in the same direction. That means \( \mathbf{u} \) and \( \mathbf{v} \) are parallel if there exists a constant \( c \) such that

Example 2: a) Using you answer from the previous example, what is \( |3\mathbf{u} - \mathbf{v}| \)?

b) Find a unit vector with the same direction as \( 3\mathbf{u} - \mathbf{v} \).

### 5.2 Displacement Vectors

**Definition:** The position vector of a point \((a_1, a_2, a_3)\), is the vector \(<a_1, a_2, a_3>\) which is the vector that starts at the origin and ends at the point \((a_1, a_2, a_3)\).

**Definition:** The displacement vector of 2 points \(U = (u_1, u_2, u_3)\) and \(V = (v_1, v_2, v_3)\) is the vector that represents the shortest distance from \(U\) to \(V\). In this case our displacement vector for \(U\) and \(V\) is

Example 3: A cat is sitting on the ground at the point \((1, 4, 0)\) watching a squirrel at the top of the tree. The tree is one unit high and its base is the point \((2, 4, 0)\). Find the displacement vectors for the following:

a) The “origin” to the cat.

b) The bottom of the tree to the squirrel.

c) The bottom of the tree to the cat.

d) From the cat to the squirrel.
Example 4: An airplane flies horizontally from west to east at 200 mph relative to the air. If it is in a steady 40 mph wind that blows southeast (45° south of east). Find the speed and direction of the plane relative to the ground.
5.3 ICE 2 – Vectors

1. Voting!: In the picture below, the unlabeled vector is closest to (a) \( \mathbf{v} + \mathbf{w} \) (b) \( \mathbf{v} \cdot \mathbf{w} \) (c) \( \mathbf{v} + 2\mathbf{w} \) (d) \( 2\mathbf{v} + \mathbf{w} \)

![Image](image.png)

2. Voting!: True or False: The vector \( \langle \frac{1}{2}, \frac{1}{2} \rangle \) is a unit vector.
   (a) True, and I am very confident
   (b) True, but I am not very confident
   (c) False, but I am not very confident
   (d) False, and I am very confident

3. Voting: Find a vector that points in the same direction as \( \langle 2, 1, 2 \rangle \), but has a magnitude of 5.
   a) \( \langle \frac{10}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{10}{\sqrt{2}} \rangle \)
   b) \( \langle \frac{10}{\sqrt{3}}, \frac{5}{\sqrt{3}}, \frac{10}{\sqrt{3}} \rangle \)
   c) \( \langle \frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}}, \frac{5}{\sqrt{3}} \rangle \)
   d) \( \langle 10, 5, 10 \rangle \)
   e) \( \langle 30, 15, 30 \rangle \)
   f) More than one of the above?

4. A plane is heading 60° north of east at an airspeed of 700 km/hr, but there is a wind blowing from the west at 50 km/hr. What is the plane’s speed relative to the ground? In what direction does the plane end up flying?
5. The vector $\mathbf{F}$ represents the force exerted on an object. It has magnitude $|\mathbf{F}|$ and a direction given by $\theta$. We denote the horizontal and vertical components of $\mathbf{F}$ as $F_x$ and $F_y$ respectively. Then $F_x = |\mathbf{F}| \cos \theta$ and $F_y = |\mathbf{F}| \sin \theta$ and the force vector is $\mathbf{F} = <F_x, F_y>$.

Suppose you pull a suitcase with a strap that makes a $60^\circ$ angle with the horizontal. The magnitude of the force you exert on the suitcase is 40 lbs.

a) Find the horizontal and vertical components of the force.

b) Is the horizontal component of the force greater if the angle of the strap is $45^\circ$ instead of $60^\circ$?

c) Is the vertical component of the force greater if the angle of the strap is $45^\circ$ instead of $60^\circ$?
### 5.4 Dot Products

So far we can add and subtract vectors and multiply them by scalars. Today we will discuss a “multiplication” for two vectors called the ________ product, denoted by ________

We use dot products to determine angles between vectors and projections.

**Definition 1 of Dot Product:** For \( \mathbf{u} = \langle u_1, u_2, u_3 \rangle \) & \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \),

\[
\mathbf{u} \cdot \mathbf{v} =
\]

Sometimes we call the dot product the ________ product. Why?

**Properties of Dot Products**

1. \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \)
2. \( \mathbf{u} \cdot \mathbf{0} = 0 \)
3. \( \mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v} \)
4. \( a(\mathbf{u} \cdot \mathbf{v}) = (a\mathbf{u}) \cdot \mathbf{v} \)
5. \( \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \)

**Definition 2 of Dot Product:** For nonzero vectors \( \mathbf{u} \) & \( \mathbf{v} \), \( \mathbf{u} \cdot \mathbf{v} = \)

where \( \theta \) is the smallest non-negative angle between \( \mathbf{u} \) & \( \mathbf{v} \). That is, \( 0 \leq \theta \leq \pi \).

Notice that if \( \mathbf{u} = 0 \) or \( \mathbf{v} = 0 \), then \( \mathbf{u} \cdot \mathbf{v} = \___________ \) and \( \theta \) is ________________.

**Note:** This definition is useful for determining angles between vectors.

**Example 1:** Let \( \mathbf{u} = \langle 2, 7, 9 \rangle \), \( \mathbf{v} = \langle -1, 7, 0 \rangle \).

a) What is \( \mathbf{u} \cdot \mathbf{v} \)?

b) What is \( \mathbf{v} \cdot \mathbf{v} \)?

**Voting Question 1:** The only way that \( \mathbf{u} \cdot \mathbf{v} = 0 \) is if \( \mathbf{u} = 0 \) or \( \mathbf{v} = 0 \).

(a) True, and I am very confident

(b) True, but I am not very confident

(c) False, but I am not very confident

(d) False, and I am very confident
**Definition:** Two vectors $\mathbf{u}$ & $\mathbf{v}$ are said to be ________ if $\mathbf{u} \cdot \mathbf{v} = ______$. That is, $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

**Definition:** A vector that is perpendicular to a plane is called a ______ vector for the plane.

**Voting Question 2:** The zero vector, $\mathbf{0}$ is orthogonal to all other vectors.
(a) True, and I am very confident
(b) True, but I am not very confident
(c) False, but I am not very confident
(d) False, and I am very confident

**Example 2:** Find the angle made by ABC where $A = (1, -3, -2)$, $B = (2, 0, -4)$, and $C = (2, 1, -4)$.

**5.4.1 Projections**

Projections tell us how much of a given vector lies in the direction of another vector.

**Definition:** The *orthogonal projection* of $\mathbf{u}$ onto $\mathbf{v}$ where $\mathbf{v} \neq 0$ is $\text{Proj}_\mathbf{v} \mathbf{u}$

Which part denotes the length of the projection?
Which part denotes the direction?
**Definition:** The *scalar projection* of $u$ onto $v$ where $v \neq 0$ is $\text{Scal}_v u = \frac{u \cdot v}{\|v\|}$

Using our properties of dot products, we see that $\text{Scal}_v u = \frac{u \cdot v}{\|v\|}$

Some texts called the scalar projection the *scalar component or $u$ in the direction of $v$*.

Now we have another definition for $\text{Proj}_v u$.

**Definition:** The work done by a constant force $F$ in moving an object along the line from $A$ to $B$ is the magnitude of the force in the direction of the motion multiplied by the distance moved. If $D$ is the displacement vector from $A$ to $B$, then the work done is $\text{Scal}_D F \|D\| = \frac{F \cdot D}{\|D\|}$.

That is, the work is $\text{Scal}_D F \|D\| = \frac{F \cdot D}{\|D\|}$.

**Example 4:** Given $u = \langle 2, 7, 9 \rangle$, $v = \langle -1, 7, 0 \rangle$ as in Example 1, what is $\text{Proj}_v u$? (Hint: Use your answers from Example 1!)
5.5 Cross Products

Last class we talked about the scalar product of two vectors. Today we will discuss the vector product. For this product when we “multiply” two vectors we get a ________

We use cross products to determine normal vectors, torque, magnetic force, and well yeah we gotta know how to take cross products.

**Definition 1 of Cross Product:** For nonzero vectors $\mathbf{u} \& \mathbf{v}$, $\mathbf{u} \times \mathbf{v} =

where $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{u} \& \mathbf{v}$ and whose direction is determined by the ________ ________ ________.

**Note:** $||\mathbf{u} \times \mathbf{v}|| =$

**Area of a Parallelogram:** The area of the parallelogram made from the 2 sides of $\mathbf{u} \& \mathbf{v}$ is ________

**Volume of a Parallelepiped:** The volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is

**Definition 2 of Cross Product:** For $\mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \& \mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$, $\mathbf{u} \times \mathbf{v} = $
Properties of Cross Products

1. \( u \times v = -(v \times u) \)  
2. \( u \times 0 = 0 \times u = 0 \)  
3. \( u \times u = 0 \)  
4. \( u \cdot (v \times w) = (u \times v) \cdot w \)  
5. \( u \times (v \times w) = (u \cdot w)v - (u \cdot v)w \)  
6. \( k(u \times v) = (ku) \times v = u \times (kv) \)

Proveable Facts:

\( i \times j = \quad j \times k = \quad k \times i = \)

Example 1: Confirm that \( i \times j = k \).

Example 2: Find a vector normal to \( u = <1, 2, 3> \) & \( v = <2, -5, -3> \).

Theorem 1: For \( u = u_1i + u_2j + u_3k \) & \( v = v_1i + v_2j + v_3k \), \( u \cdot (u \times v) \) and \( v \cdot (u \times v) \) are both equal to ________

“Proof”: 
Theorem 2: \( \mathbf{u} \) \& \( \mathbf{v} \) in a 3-Space are parallel if and only if \( \mathbf{u} \times \mathbf{v} = \) __________

Notice this is easy to see from our angle definition of the cross product since two vectors are parallel if and only if the angle between them is either 0° or 180° which means \( \sin \theta = \) __________.

5.5.1 Torque

When we are using a wrench, the “twisting” power you generate depends on 3 things

1. 
2. 
3. 

Definition: Torque is the a twisting generated by a force acting at a distance from a pivot point.

If a force \( \mathbf{F} \) is applied to a point \( P \) at the head of a vector \( \mathbf{r} = \overrightarrow{OP} \) the torque, \( \mathbf{\tau} = \) __________

\[ ||\mathbf{\tau}|| = \]

where \( \theta \) is the angle between \( \mathbf{r} \) \& \( \mathbf{F} \).

Example 3: Let \( \mathbf{r} = \overrightarrow{OP} = 1, -1, 2 \) > and suppose a force \( \mathbf{F} = 10, 10, 0 \) > is applied to \( P \). Find the torque about \( O \) that is produced.
5.6 ICE 3 –Dot and Cross Products

Cross and Dot products

1. Consider the vectors \( \mathbf{u} = 2\mathbf{i} \) and \( \mathbf{v} = 3\mathbf{j} \)

   a) Draw the two vectors.

   b) Find the magnitude of \( \mathbf{u} \times \mathbf{v} \)

   c) Determine the direction of \( \mathbf{u} \times \mathbf{v} \)

   d) Verify your previous answers by calculating \( \mathbf{u} \times \mathbf{v} \) in component form.

   e) Find the magnitude of \( \mathbf{v} \times \mathbf{u} \)

   f) Determine the direction of \( \mathbf{v} \times \mathbf{u} \)

   g) Verify your previous answer by calculating \( \mathbf{v} \times \mathbf{u} \) in component form.

2. Voting!: Two vectors have a dot product of 14. To guarantee the dot product is equal to 28, you could:
   (a) double the angle between the vectors
   (b) double the length of both vectors
   (c) double the length of one vector
   (d) none of the above

3. Voting!: When vectors are closely aligned, is \( \text{Scal}_\mathbf{v} \mathbf{u} \) large or small?
   (a) Large, and I am very confident
   (b) Large, but I am not very confident
   (c) Small, but I am not very confident
   (d) Small, and I am very confident
4. Voting!: What is \( \text{Proj}_v u \) when \( v \) & \( u \) are orthogonal?
   
   (a) \( v \)
   (b) undefined
   (c) 0
   (d) \(-v\)

5. A vector that is normal to the plane containing the vectors \( a = 4i - j + 2k \) and \( \vec{b} = -i + 5j + 3k \) is

   (a) \(-13i + 14j + 19k\)
   (b) \(13i + 14j - 19k\)
   (c) \(-13i - 14j + 19k\)
   (d) \(13i - 14j - 19k\)
   (e) More than one of the above.

6. If \( \vec{d} = \vec{a} \times \vec{b} \), then \( \vec{a} \cdot \vec{d} = \)

   (a) \( \vec{a} \times (\vec{b} \cdot \vec{b}) \)
   (b) 0
   (c) \( \vec{a} \times (\vec{a} \cdot \vec{b}) \)
   (d) \((\vec{a} \cdot \vec{b}) \times \vec{b}\)
6 Vector Valued Functions

Definition: A vector-valued function \( \mathbf{F}(t) \) associates each \( t \) with a vector: 
\[
\mathbf{F}(t) =
\]

Note: Now there is a difference now between \( f(t) \) and \( f(t) \)

Since \( \mathbf{F}(t) \) is a vector, we can do what operations to vector valued functions?

So now we just now have functions of \( t \) rather than just numbers.

And, since \( \mathbf{F}(t) \) is a function, we also want to discuss calculus ideas in terms of \( \mathbf{F}(t) \).

Calculus for Vector Valued Functions: For \( \mathbf{F}(t) = < f(t), g(t), h(t) > \), 
\[
\mathbf{F}'(t) =
\]
\[
\int \mathbf{F}(t) dt =
\]
\[
\lim_{t \to a} \mathbf{F}(t) dt =
\]
\[
\frac{d}{dt} [\mathbf{F}(t) + \mathbf{G}(t)] =
\]
\[
\frac{d}{dt} [c\mathbf{F}(t)] =
\]
\[
\frac{d}{dt} [p(t)\mathbf{F}(t)] =
\]
\[
\frac{d}{dt} [\mathbf{F}(t) \cdot \mathbf{G}(t)] =
\]
\[
\frac{d}{dt} [\mathbf{F}(t) \times \mathbf{G}(t)] =
\]
\[
\frac{d}{dt} [\mathbf{F}(p(t))] =
\]
Example 1: Let \( \mathbf{a}(t) = < 2t, 0, e^{-t} > \) and \( \mathbf{b}(t) = < \cos t, 5t, 1 > \). Determine \( \frac{d}{dt} [\mathbf{a}(t) \cdot \mathbf{b}(t)] \).

Voting Question 1: Could I have done the dot product first and then differentiated term by term?
   a) Yes and I am very confident
   b) Yes, but I am not very confident
   c) No and I am very confident
   d) No, but I am not very confident
   e) I love cats.

Voting Question 2: Does order matter in the derivative of the dot product?
   a) Yes and I am very confident
   b) Yes, but I am not very confident
   c) No and I am very confident
   d) No, but I am not very confident
   e) I hate math.

Voting Question 3: Does order matter in the derivative of the cross product?
   a) Yes and I am very confident
   b) Yes, but I am not very confident
   c) No and I am very confident
   d) No, but I am not very confident
   e) Will this class ever end?

Example 2: Let \( \mathbf{a}(t) = < 2t, 0, e^{-t} > \) and \( f(t) = 7t \). Determine \( \frac{d}{dt} [f(t)\mathbf{a}(t)] \).
Example 3: Let \( \mathbf{a}(t) = \langle 2t, 0, e^{-t} \rangle \) and \( \mathbf{b}(t) = \langle \cos t, 5t, 1 \rangle \). Determine \( \frac{d}{dt}[\mathbf{a}(t) \times \mathbf{b}(t)] \).

Definition: Given a position vector \( \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle \),
the velocity vector \( \mathbf{v}(t) = \)
the acceleration vector \( \mathbf{a}(t) = \)
And the speed is given by

Example 4: Consider the trajectory given by the position vector \( \mathbf{p}(t) = \langle e^{-t} \cos t, e^{-t} \sin t, 2 - 2e^{-t} \rangle \). for \( t \geq 0 \).

a) What is the initial point of the trajectory?

b) What is the terminal point of the trajectory?
c) What is the initial velocity? What is the initial speed?

**Just for The Cool Cats:** When is the speed the greatest? This is long, but interesting. =)
7 Lines and Planes

7.1 Lines In Space!

What determines a line?

**Representation of a Line:** Let \( \vec{v} = \langle a, b, c \rangle \) be a direction vector for a line and let \( \vec{r}_0 = \langle x_0, y_0, z_0 \rangle \) be the position vector for a point \( P_0 = (x_0, y_0, z_0) \) on the same line.

**Vector Representation of a Line:** \( \vec{r} = \)

**Parametric Representation of a Line:**
\[
\begin{align*}
x &= \\
y &= \\
z &=
\end{align*}
\]

**Symmetric Representation of a Line:**

Note: If \( a, b, \) or \( c = 0 \), we can still eliminate \( t \). For example if \( b = 0 \), then we have

**Example 1:** Find an equation of a line that contains \( u = (2, 7, 9) \) and is parallel to the vector \( < 3, -1, 7 > \).

**Example 2:** Find an equation of a line that contains \( u = (2, 7, 9) \) and \( v = (1, -1, 1) \).
Example 3: Find the symmetric equation of the line through $(-5, 7, -3)$ that is perpendicular to both $<2, 1, -3>$ and $<5, 4, -1>$.

7.2 Planes

We already mentioned that an equation of a plane is in the form $Ax + By + Cz = D$. Now we will discuss another way to describe a plane.

Recall a line in the $xy$ plane can be described with a _______ and a _______.

A plane can be described by a _______ and a _______ vector.

**Standard Form of Plane:** The equation of the plane going through the point $P = (x_1, y_1, z_1)$ with a normal vector $\mathbf{n} = <a, b, c>$ can be written as

Useful fact: Planes $Ax + By + Cz = D$ and $ax + by + cz = d$ are parallel if there exists a constant $k$ such that $<A, B, C> = k <a, b, c>$. The planes are identical if they are parallel and $D = kd$.

Example 4: a) Find the equation of the plane through $(0,0,1)$ and perpendicular to $2i+4j+k$.

b) Find the equation of the plane through $(-1, 2, 5)$ and parallel to the plane you found in part a.
7.3 Tangent Lines

If a curve, \( \mathbf{r} \) has the position vector \( \langle f(t), g(t), h(t) \rangle \), the tangent line to \( \mathbf{r} \) has the direction vector:

There is exactly _______ plane perpendicular to a smooth curve at any given point. If we have a direction vector for the tangent line to the curve at \( P \), then it is a _______ vector for the plane.

Example 5: Find the equation of the plane perpendicular to the curve \( \mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle \) at \( t = \frac{\pi}{2} \).
7.4 ICE 4 – Vector Functions

Consider the curve traced by the vector valued function \( \mathbf{r}(t) = \langle \sin t \cos t, \cos^2 t, \sin t \rangle \).

1. Show that the curve lives on the surface of the unit sphere \( x^2 + y^2 + z^2 = 1 \).

2. Compute \( |\mathbf{r}(t)| \). [Hint: this should now be easy!]

3. Compute \( \mathbf{r}'(t) \).

4. Show that \( \mathbf{r}'(t) \) is perpendicular to \( \mathbf{r}(t) \) for all \( t \). (You can verify this analytically. Also this is a general result, true whenever \( |\mathbf{r}(t)| \) is constant. ) You can verify analytically on the back of the page after completing 5 and 6.

5. Find the tangent line to the curve at \( t = 0 \).
6. If a rocket were flying along the curve traced by \( \mathbf{r}(t) \) and suddenly turned off its engines at \( t = 0 \) where would it be when \( t = 2 \)?

7. Verify analytically that \( \mathbf{r}'(t) \) is perpendicular to \( \mathbf{r}(t) \) for all \( t \).
8 Curvature

8.1 Definitions of Curvature

There are 2 ways to change velocity. We can change our _________ or change our _________.

Intuitively, curvature measures how much a curve bends at a point.

Given a position vector, \( \vec{r} \), what is our tangent vector?

Let \( T(t) \) represent the unit tangent vector for \( \vec{r} \). So \( T(t) = \)

So we can define curvature as \( \kappa = \left\| \frac{dT}{ds} \right\| \)

Note we are differentiating with respect to \( s \) which represents __________ __________. This definition is not very helpful though. Instead we can do some manipulation (see text) and arrive at the following definition for curvature:

**Definition 1:** The curvature, \( \kappa \), of curve \( r(t) \) is \( \kappa(t) = \)

**Alternative Definition 2:** \( \kappa(t) = \)

**Alternative Definition 3:** Given \( r(t) =< f(t), g(t) > \), \( \kappa(t) = \)

**Example 1:** Find the curvature of \( r(t) =< t, \ln(\cos(t)) > \) using Definition 1.
Example 2: Find the curvature of \( \mathbf{r}(t) = \langle t, \ln(\cos(t)) \rangle \) using Definition 2.

Example 3: Find the curvature of \( \mathbf{r}(t) = \langle t, \ln(\cos(t)) \rangle \) using Definition 3.

8.2 Components of Acceleration

Let \( \mathbf{T}(t) \) represents the unit tangent vector for \( \mathbf{r} \). Thus \( \mathbf{T}(t) \cdot \mathbf{T}(t) = \) ________ and thus

So \( \mathbf{T}(t) \cdot \mathbf{T}'(t) = \) ________ \( \Rightarrow \mathbf{T}(t) \) is ________ with \( \mathbf{T}'(t) \).

**Definition:** We define this normal vector (after normalizing it) as the unit normal vector \( \mathbf{N}(t) = \) ________

Note that there are infinitely many unit vectors perpendicular to \( \mathbf{T} \) at a point. \( \mathbf{N}(t) \) is the principle unit normal vector and is normal to \( \mathbf{T}(t) \) and points in the direction of “curving”.
One other important unit vector is our binormal vector which is orthogonal to both \( \mathbf{T} \) & \( \mathbf{N} \).

**Definition:** The unit binormal vector \( \mathbf{B}(t) = \)

**Example 4:** Determine the unit vectors \( \mathbf{T}, \mathbf{N} \) and \( \mathbf{B} \) for \( \mathbf{r}(t) = < t, \ln(\cos(t)) > \).

When we are in a car which is accelerating forward, we feel pushed _______ and if we turn right, we feel pushed to the _______. These two kinds of acceleration are called the *tangential and normal components of acceleration*.

We can write our \( \mathbf{a}(t) \) in terms of these components. That is, we can write \( \mathbf{a}(t) = \)

We can determine the scalars \( a_T \) and \( a_N \) by:

**Definitions of Components:**

\[
 a_T = \quad a_N =
\]

**Example 5:** Determine the components of acceleration for \( \mathbf{r}(t) = < t, \ln(\cos(t)) > \) at \( t = 0 \).
8.3 ICE – Curvature

2 sides!

1. The curve $\mathbf{r}(t) = <2\cos t, 2\sin t, 3t>$ is shown below. Find $\mathbf{T}, \mathbf{N}, \& \mathbf{B}$. Sketch $\mathbf{T}, \mathbf{N}, \& \mathbf{B}$ when $t = \frac{3\pi}{2}$. 
2. Below is the graph of the space curve given by the vector valued equation 
\[ \mathbf{r}(t) = 2t \mathbf{i} + t^2 \mathbf{j} - \frac{1}{3} t^3 \mathbf{k}. \]

a) Based on an inspection of the graph, at what point is the curvature the greatest?

b) Find the curvature function \( \kappa(t) \) for the space curve.

c) Verify analytically the point at which the curvature is the greatest.
9 Surfaces

9.1 Review of Conic Sections

9.1.1 Parabolas

What is a parabola?

Example 1: Sketch the graph of \( x^2 = -8y \).
### Parabola Notes

<table>
<thead>
<tr>
<th>Function</th>
<th>Equation</th>
<th>Vertex</th>
<th>Focus</th>
<th>Directrix</th>
<th>Opens</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Vertical</td>
<td>$x^2 = 4py$</td>
<td>(0, 0)</td>
<td>(0, $p$)</td>
<td>$y = -p$</td>
<td>up if $p &gt; 0$ down if $p &lt; 0$</td>
<td></td>
</tr>
<tr>
<td>General Vertical</td>
<td>$(x - h)^2 = 4p(y - k)$</td>
<td>$(h, k)$</td>
<td>$(h, k + p)$</td>
<td>$y = k - p$</td>
<td>up if $p &gt; 0$ down if $p &lt; 0$</td>
<td></td>
</tr>
<tr>
<td>Standard Horizontal</td>
<td>$y^2 = 4px$</td>
<td>(0, 0)</td>
<td>(0, 0)</td>
<td>$x = -p$</td>
<td>right if $p &gt; 0$ left if $p &lt; 0$</td>
<td></td>
</tr>
<tr>
<td>General Horizontal</td>
<td>$(y - k)^2 = 4p(x - h)$</td>
<td>$(h, k)$</td>
<td>$(h + p, k)$</td>
<td>$x = h - p$</td>
<td>right if $p &gt; 0$ left if $p &lt; 0$</td>
<td></td>
</tr>
</tbody>
</table>

**Voting Question 1:** Find an equation of the parabola that has vertex (2, 1) and directrix $y = 5$.

(a) $(x - 2)^2 = 20(y - 1)$

(b) $(x - 2)^2 = -20(y - 1)$

(c) $(y - 1)^2 = 20(x - 2)$

(d) $(y - 1)^2 = -20(x - 2)$

### 9.1.2 Ellipses and Hyperbolas

What is an ellipse?

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## Ellipse Notes

<table>
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<tr>
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<th>Equation</th>
<th>Center</th>
<th>Foci</th>
<th>Vertices</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Standard Horizontal (Wide)</strong></td>
<td>$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  [ a &gt; b ]  [ c^2 = a^2 - b^2 ]</td>
<td>(0, 0)</td>
<td>(±c, 0)</td>
<td>major: (±a, 0)  minor: (0, ±b)</td>
<td>$y$</td>
</tr>
<tr>
<td><strong>General Horizontal (Wide)</strong></td>
<td>$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$  [ a &gt; b ]  [ c^2 = a^2 - b^2 ]</td>
<td>(h, k)</td>
<td>(h ± c, k)</td>
<td>minor: (h, k ± b)  major: (h ± a, k)</td>
<td>$y$</td>
</tr>
<tr>
<td><strong>Standard Vertical (Tall)</strong></td>
<td>$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$  [ a &gt; b ]  [ c^2 = a^2 - b^2 ]</td>
<td>(0, 0)</td>
<td>(0, ±c)</td>
<td>minor: (±b, 0)  major: (0, ±a)</td>
<td>$y$</td>
</tr>
<tr>
<td><strong>General Vertical (Tall)</strong></td>
<td>$\frac{(y - k)^2}{a^2} + \frac{(x - h)^2}{b^2} = 1$  [ a &gt; b ]  [ c^2 = a^2 - b^2 ]</td>
<td>(h, k)</td>
<td>(h, k ± c)</td>
<td>minor: (h ± b, k)  major: (h, k ± a)</td>
<td>$y$</td>
</tr>
</tbody>
</table>

What is a hyperbola?

![Hyperbola Graph](image)
<table>
<thead>
<tr>
<th>Function</th>
<th>Equation</th>
<th>Center</th>
<th>Foci</th>
<th>Vertices</th>
<th>Asymptotes</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard Vertical (Tall)</td>
<td>( \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 )  ( \frac{c^2}{a^2 + b^2} )</td>
<td>(0,0)</td>
<td>(0,±c)</td>
<td>(0,±b) (±a,0)</td>
<td>( y = \pm \frac{a}{b}x )</td>
<td></td>
</tr>
<tr>
<td>General Vertical (Tall)</td>
<td>( \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 ) ( \frac{c^2}{a^2 + b^2} )</td>
<td>(h,k)</td>
<td>(h,k±c)</td>
<td>(h,k±b) (h±a,k)</td>
<td>( y = \pm \frac{a}{b}(x-h)+k )</td>
<td></td>
</tr>
<tr>
<td>Standard Horizontal (Wide)</td>
<td>( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 )  ( \frac{c^2}{a^2 + b^2} )</td>
<td>(0,0)</td>
<td>(±c,0)</td>
<td>(±a,0)</td>
<td>( y = \pm \frac{b}{a}x )</td>
<td></td>
</tr>
<tr>
<td>General Horizontal (Wide)</td>
<td>( \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 ) ( \frac{c^2}{a^2 + b^2} )</td>
<td>(h,k)</td>
<td>(h±c,k)</td>
<td>(h±a,k)</td>
<td>( y = \pm \frac{b}{a}(x-h)+k )</td>
<td></td>
</tr>
</tbody>
</table>

Given a conic sections of the form: \( Ax^2 + Cy^2 + Dx + Ey + F = 0 \), we have a...

Circle if: Hyperbola if:

Ellipse if: Parabola if:

**Example 2:** What type of conic section is \( 4x^2 - 9y^2 + 16x + 18y = 29 \)?
Another way to define these conic sections is by a fixed positive constant $e$ called the *eccentricity* for all points $P$,

$$e = \frac{|PF|}{Pl}$$

Where $F$ is a fixed point called the *focus* and $l$ is a fixed line called the *directrix* in a plane.

We have an ellipse if $e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$

**Voting Question 2:** Find the vertices/major axis and the endpoints of the minor axis for the ellipse given by the equation $9x^2 + 4y^2 = 16$.

(a) vertices: $(\pm 2, 0)$; endpoints of minor axis: $(0, \pm 4/3)$
(b) vertices: $(0, \pm 2)$; endpoints of minor axis: $(\pm 4/3, 0)$
(c) vertices: $(\pm 2, 0)$; endpoints of minor axis: $(0, \pm 3/4)$
(d) vertices: $(0, \pm 2)$; endpoints of minor axis: $(\pm 3/4, 0)$

**Example 3:** Sketch the graph of $y^2 - 4x^2 - 2y + 16x = -1$. 

![Graph of $y^2 - 4x^2 - 2y + 16x = -1$.](image)
### 9.1.3 Optional Topic: Translations and Rotations

We know how to deal with Conic Sections in the form: \( Ax^2 + Cy^2 + Dx + Ey + F = 0. \) But what if we have an \( xy \) term like \( 4x^2 - 3xy = 18? \)

In Example 3, we wrote \( y^2 - 4x^2 - 2y + 16x = -1 \) as \( \frac{(x + 2)^2}{4} - \frac{(y - 1)^2}{16} = 1 \) by completing the square.

We can think of this as a ________ of axes by setting new coordinates \( u = \________ \) and \( v = \________. \) The \( u-v \) axes are ________ and ________ to the \( x-y \) axes.

If we have \( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \) as our form, this is a ________ of the \( x-y \) axes.

By Triangle 1: \( \cos(\theta + \phi) = \frac{x}{r} \)
\[
x = r \cos(\theta + \phi) = r(\cos \theta \cos \phi - \sin \theta \sin \phi)
\]
\[
= (r \cos \phi) \cos \theta - (r \sin \phi) \sin \theta
\]

By Triangle 2: \( \cos \phi = \frac{u}{r} \Rightarrow u = r \cos \phi \)
\( \sin \phi = \frac{v}{r} \Rightarrow v = r \sin \phi \)

Thus
\[
x = \________
\]
\[
y = \________
\]

How do we determine the angle \( \theta? \)

\( Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \)

\( \cot(2\theta) = \________ \)

Then we just solve for \( \theta! \)
Example 4: Eliminate the cross-product term for $2xy - 1 = 0$ and put it in terms of $u$ and $v$ to recognize the conic section.
9.2 Graphing Surfaces

Today, we will work on graphing three dimensional surfaces. When graphing surfaces in a 3-Space, it often helpful to look at good cross sections -especially traces!

**Definition:** A *cross section* of a surface is the set of points at which the surface intersects a plane that is parallel to one of the coordinate planes.

* are cross sections of the $xy$, $xz$, and $yz$ planes.

It may also be useful to memorize basic surfaces. But even if you do this, it is good to know how to check if you have the correct curve. Let’s start with some basic examples of surfaces.

**Example 1:** Provide a rough sketch and describe in words the surfaces described by the following equations. Key: Think about the surfaces and traces before you calculate anything.

a) $x^2 + y^2 + z^2 = 25$

![Graph of $x^2 + y^2 + z^2 = 25$](image1)

b) $x^2 + y^2 = 3$

![Graph of $x^2 + y^2 = 3$](image2)
c) $y^2 + z^2 = 1$

Notice in Example 1b and 1c, we were missing one of our three variables. These are examples of a family of functions called \underline{cylinder}.

**Definition:** Given a curve $C$ in a plane $P$ and a line $l$ not in $P$, a cylinder is the surface consisting of all lines parallel to $l$ that pass through $C$.

**Steps for Sketching Cylinders**

1) Graph the curve in the trace $x = 0, y = 0, \text{ or } z = 0$ depending on which variable is missing.
2) Draw that trace on your graph.
3) Draw a second trace (a copy of the curve in step 2) in a plane parallel to the trace.
4) Draw lines parallel to the $x, y, \text{ or } z$ axis passing through the two traces (this is determined by our missing variable).

**Example 2:** Sketch the curve $x - \sin(z) = 0$. 
Some Guidelines for Sketching:

- Determine **Intercepts** (where the surfaces intersect coordinate axes).
- Determine any constraints on variables (for example: \( x > 0 \)).
- Sketch the **Traces** or a good **cross section** (by setting one variable to 0 or a constant).
- Sketch at least two traces in parallel planes (for example traces with \( z = 0, z = \pm 1 \)).
- Put it together in a Practice Sketch.
- Sketch it, sketch it real good!

**Example 3:** Sketch the curve \( x = \frac{y^2}{4} + \frac{z^2}{9} \).
9.3 General Forms of Surfaces

Ellipsoid: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \)

Hyperboloid of 1 sheet:
\[
\begin{align*}
\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} &= 1, & \frac{y^2}{a^2} + \frac{z^2}{b^2} - \frac{x^2}{c^2} &= 1,
\end{align*}
\]

Hyperboloid of 2 sheets:
\[
\begin{align*}
\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} &= 1, & \frac{y^2}{a^2} - \frac{z^2}{b^2} - \frac{x^2}{c^2} &= 1,
\end{align*}
\]

Elliptic Paraboloid:
\[
\begin{align*}
z &= \frac{x^2}{a^2} + \frac{y^2}{b^2}, & y &= \frac{x^2}{a^2} + \frac{z^2}{c^2}, & x &= \frac{y^2}{a^2} + \frac{z^2}{b^2}
\end{align*}
\]

Hyperbolic Paraboloid:
\[
\begin{align*}
z &= \frac{y^2}{a^2} - \frac{x^2}{b^2}, & z &= \frac{x^2}{a^2} - \frac{y^2}{b^2}, & y &= \frac{a^2}{b^2} - \frac{x^2}{z^2}, & x &= \frac{a^2}{b^2} - \frac{y^2}{z^2}
\end{align*}
\]

Elliptic Cone:
\[
\begin{align*}
x^2 + \frac{y^2}{b^2} - \frac{z^2}{c^2} &= 0, & \frac{x^2}{a^2} + \frac{z^2}{b^2} - \frac{y^2}{c^2} &= 0, & \frac{y^2}{a^2} + \frac{z^2}{b^2} - \frac{x^2}{c^2} &= 0
\end{align*}
\]
9.4 ICE 6 – Surfaces

Match the graphs with the equations. Don’t just look at your general forms, try to think about it using traces and cross sections. Then check with your general forms.

a) $y - z^2 = 0$

b) $2x + 3y - z = 5$

c) $4x^2 + \frac{y^2}{2} + z^2 = 9$

d) $x^2 + \frac{y^2}{2} - z^2 = 9$

e) $x^2 + \frac{y^2}{2} = z^2$

f) $x^2 + \frac{z^2}{2} - y^2 = 9$

g) $z^2 + \frac{y^2}{2} = x^2$

h) $z = x^2 - y^2$

i) $9 = z^2 + y^2$
10 Multivariate Functions

10.1 Functions of Two Variables

So far, we have discussed real valued (scalar) functions and vector valued functions so far in this class. Today we introduce multivariate functions and in particular functions of two variables.

Notation: Functions of two variables are usually denoted explicitly as $z = \underline{}$ or $\underline{} = 0$.

Recall the domain of a function (unless specified) is the set of all input points $(x, y)$ for which our function makes sense. The range of a function is the set of all real numbers $z$ that are reached as our input points vary.

Example 1: What is the domain of $f(x, y) = \frac{\ln(y^2 - x)}{x - 2}$?

When we are asked to determine the graph of $f(x, y)$, we consider the surface ______. What is special about these surfaces is that we have a function, so each $(x, y)$ corresponds with _______ _______ $z$. In other words, each line perpendicular to the $xy$ plane intersects the surface at most _______ time.

Voting Question 1: Could we ever get an ellipsoid as the graph of a function of two variables?

(a) Yes and I am very confident.  
(b) Yes, but I am not very confident. 
(c) No and I am very confident. 
(d) No, but I am not very confident.

Example 2: Sketch the graph of $f(x, y) = \sqrt{4 + x^2 + y^2}$
10.2 Level Curves and Contour Maps

Sometimes sketching $f(x, y)$ in $\mathbb{R}^3$ makes people sad. Another way to represent or picture our surfaces are _______ maps.

Definition: A level curve of the function $f(x, y)$ is the curve with the equation _______

The set of level curves is called a _______ map.

Notice that when the level curves are closer together, the graph is _______.

And when the level curves are further apart, the graph is _______.

Examples: $f(x, y) = 9 - \sqrt{4 + x^2 + y^2}$ and $g(x, y) = \sqrt{x^2 - \frac{y^2}{2}}$
Contours for $f(x, y) = 9 - \sqrt{4 + x^2 + y^2}$ and $g(x, y) = \sqrt{x^2 - \frac{y^2}{2}}$.

Example 3: Draw the contour map for $f(x, y) = x^2 + y^2$
10.3  ICE 7 –Level Curves

For each of the following functions numbered 1-6, draw a contour map of the function by sketching the level curves for $k = 0, 1, 2, 3, 4$. Then match the contour maps with the associated graphs I-VI from the other side of the sheet.

1. $f(x, y) = x - y$ Graph _______

2. $f(x, y) = x^2 - y$ Graph _______

3. $f(x, y) = \sqrt{x^2 + y^2}$ Graph _______

4. $f(x, y) = xy$ Graph _______

Turn page over for more.
5. \( f(x, y) = x^2 + y^2 \) Graph

6. \( f(x, y) = \sqrt{9 - x^2 - y^2} \) Graph
11 Partial Derivatives

**Definition:** Given $f(x, y)$, the partial derivative with respect to $x$ at $(a, b)$ is $f_x(a, b) = \ldots$

In general the $x$ partial derivative is denoted by ________ or ________.

**Key Idea:** To find the partial derivative with respect to $x$, $f_x$, treat ________ as a constant and differentiate $f(x, y)$ with respect to ________.

Similarly we can find the partial derivative with respect to $y$ by treating ________ as a constant and differentiate $f(x, y)$ with respect to ________.

**Example 1:** Find $f_x$ and $f_y$ for $f(x, y) = xe^{2y}$

---

1 from [https://www.reddit.com/r/funny/comments/1lx17w/because_calculus/](https://www.reddit.com/r/funny/comments/1lx17w/because_calculus/)
Example 2: Find \( f_x(-2, -1, 8) \) for \( f(x, y, z) = \left(\frac{xy}{z}\right)^{\frac{1}{2}} \)

11.1 Higher Partial Derivatives

\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \\
\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \\
\frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \\
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} =
\]

Example 3: Find all 4 second order partial derivatives for \( f(x, y) = xe^{2y} \)

Notice \( f_{xy} = \text{________} \)! This is true all the time!*

*Well as long as \( f_{xy} \) exist and \( f_{yx} \) are continuous.
11.2 Thinking about Partial Derivatives

So far, we have described partial derivatives algebraically. Now let’s look into how we come across them in other situations.

11.2.1 Partial Derivatives Numerically

Example 4: The following table contains values for $f(x, y)$. Use it to estimate $f_x(3, 2)$ and $f_y(3, 2)$.

<table>
<thead>
<tr>
<th>$y$</th>
<th>1.8</th>
<th>2</th>
<th>2.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>12.5</td>
<td>10.2</td>
<td>9.3</td>
</tr>
<tr>
<td>3.0</td>
<td>18.1</td>
<td>17.5</td>
<td>15.9</td>
</tr>
<tr>
<td>3.5</td>
<td>20.0</td>
<td>22.4</td>
<td>26.1</td>
</tr>
</tbody>
</table>

11.2.2 Partial Derivatives Verbally

Example 5: Let $N$ be the number of applicants to a university, $p$ the charge for food and housing at the university, and $t$ the tuition. Suppose that $N$ is a function of $p$ and $t$ such that $\frac{\partial N}{\partial p} < 0$ and $\frac{\partial N}{\partial t} < 0$. What information is gained by noticing that both partials are negative?
11.2.3 Partial Derivatives Graphically

**Voting Question 1:**
At point Q in the diagram below, which of the following is true?

![Diagram of function f(x, y)](image)

(a) $f_x > 0, f_y > 0$
(b) $f_x > 0, f_y < 0$
(c) $f_x < 0, f_y > 0$
(d) $f_x < 0, f_y < 0$

**Voting Question 2:**
Using the contour plot of $f(x, y)$, which of the following is true at the point (4, 2)?

![Contour plot of f(x, y)](image)

(a) $f_x > 0$ and $f_y > 0$
(b) $f_x > 0$ and $f_y < 0$
(c) $f_x < 0$ and $f_y > 0$
(d) $f_x < 0$ and $f_y < 0$

**Voting Question 3:**
At which point above the $xy$ plane to the right will both partial derivatives be positive?

- a) $(-5, 5)$
- b) $(-5, -5)$
- c) $(5, -5)$
- d) $(5, 5)$
Understanding Higher Partials Graphically:

Case 1: $f_{xx} > 0$ or $f_{yy} > 0$: Think “_________ ________”
- $f_y$ or $f_x$ is positive, and $f$ is increasing at an increasing rate
- $f_y$ or $f_x$ is negative, and $f$ is decreasing at a decreasing rate

Case 2: $f_{xx} < 0$ or $f_{yy} < 0$: Think “_________ ________”
- $f_y$ or $f_x$ is positive, and $f$ is increasing at a decreasing rate
- $f_y$ or $f_x$ is negative, and $f$ is decreasing at an increasing rate

Case 3: $f_{xx} = 0$ or $f_{yy} = 0$: Think “no ________”
- $f_y$ or $f_x$ is positive, and $f$ is increasing at a constant rate
- $f_y$ or $f_x$ is negative, and $f$ is decreasing at a constant rate
- The point you are looking at is an “inflection point”

Case 4: Mixed Partialls: To look at $f_{xy}$ consider the rate of change of the slope in the $x$-direction as one moves in the $y$-direction.
For $f_{yx}$ consider the rate of change of the slope in the $y$-direction as one moves in the $x$-direction.

Example 6: At point Q in the diagram below, what are the signs of $f_{xx}$, $f_{yy}$, and $f_{xy}$?
11.3 Visualizing Partial Derivatives with Play-Doh
For each of the contour plots below...

a. Determine if $f_x$, $f_y$, $f_{xx}$, $f_{yy}$, and $f_{xy}$ are positive, negative, or 0 at (0,0).
b. Represent the surface with Play Doh.
c. Slice the Play Doh along $y = 0$. Sketch the edge of the new curve, on an $x - z$ axis. Determine if $f_x$ and $f_{xx}$ are positive, negative or 0 from this curve. Compare to Number 1.
d. Rebuild the surface with Play Doh. Now, slice the Play Doh along $x = 0$. Sketch the edge of the new curve, on an $y - z$ axis. Determine if $f_y$ and $f_{yy}$ are positive, negative or 0 from this curve. Compare to Number 1.
e. Choose the equation that best represents the function you are working with. Compute $f_x$, $f_y$, $f_{xx}$, $f_{yy}$, and $f_{xy}$ and evaluate them at (0,0).

- $f(x,y) = \sin(x)$
- $f(x,y) = \sin(x) + \sin(y)$
- $f(x,y) = e^y \sin(x)$
- $f(x,y) = x^2 - y^2$
- $f(x,y) = x + 2y$

1. 

![Contour Plot](image-url)
After you finish all of these, look at the collection of surface graphs, and determine which one matches each of the above functions.
Surfaces
Play Doh Assignment

Directions: For each Contour plot, answer questions a-e.

Contour Plot 1:
a) From plot, I think $f_x$ is ________, $f_y$ is ________, $f_{xx}$ is ________, $f_{yy}$ is ________, and $f_{xy}$ is ________.

b) Represent the surface with Play Doh using the contour plot. Then pick which graph you think your contour plot/Play Doh surface represents. Attach picture of surface to your document that contains all pictures.

c) From my slice, Here are my my graphs on the xz and yz axes:

\[
\begin{array}{c}
\text{z} \\
\hdashline
\text{x} \\
\hdashline
\text{y} \\
\text{z}
\end{array}
\]

I think $f_x$ is ________ and $f_{xx}$ is ________. Attach picture of slice to your document that contains all pictures.

d) From my slice, I think $f_y$ is ________ and $f_{yy}$ is ________. Attach picture of slice to your document that contains all pictures.

e) Pick the equation you think represents this function and determine the signs of your partial derivatives algebraically. I think my function is ________________. From this, I found that $f_x$ is ________, $f_y$ is ________, $f_{xx}$ is ________, $f_{yy}$ is ________, and $f_{xy}$ is ________.

f) Did all of your choices for your partial derivatives agree? Why or why not, can you determine where you made a mistake if you made one? Did slicing help you determine the signs of your partial derivatives?
Contour Plot 2:
a) From plot, I think $f_x$ is ________, $f_y$ is ________, $f_{xx}$ is ________, $f_{yy}$ is ________, and $f_{xy}$ is ________.

b) Represent the surface with Play Doh using the contour plot. Then pick which graph you think your contour plot/Play Doh surface represents. Attach picture of surface to your document that contains all pictures.

c) From my slice, Here are my my graphs on the xz and yz axes:

\[
\begin{array}{c}
\text{z} \\
\hline
\hline
\text{x} \\
\hline
\hline
\text{y}
\end{array}
\]

I think $f_x$ is ________ and $f_{xx}$ is ________. Attach picture of slice to your document that contains all pictures.

d) From my slice, I think $f_y$ is ________ and $f_{yy}$ is ________. Attach picture of slice to your document that contains all pictures.

e) Pick the equation you think represents this function and determine the signs of your partial derivatives algebraically. I think my function is ______________. From this, I found that $f_x$ is ________, $f_y$ is ________, $f_{xx}$ is ________, $f_{yy}$ is ________, and $f_{xy}$ is ________.

f) Did all of your choices for your partial derivatives agree? Why or why not, can you determine where you made a mistake if you made one? Did slicing help you determine the signs of your partial derivatives?
Contour Plot 3:

a) From plot, I think $f_x$ is __________, $f_y$ is __________, $f_{xx}$ is __________, $f_{yy}$ is __________, and $f_{xy}$ is __________.

b) Represent the surface with Play Doh using the contour plot. Then pick which graph you think your contour plot/Play Doh surface represents. Attach picture of surface to your document that contains all pictures.

c) From my slice, Here are my my graphs on the xz and yz axes:

\[ z \quad | \quad x \quad | \quad y \]

I think $f_x$ is __________ and $f_{xx}$ is __________. Attach picture of slice to your document that contains all pictures.

d) From my slice, I think $f_y$ is __________ and $f_{yy}$ is __________. Attach picture of slice to your document that contains all pictures.

e) Pick the equation you think represents this function and determine the signs of your partial derivatives algebraically. I think my function is ______________. From this, I found that $f_x$ is __________, $f_y$ is __________, $f_{xx}$ is __________, $f_{yy}$ is __________, and $f_{xy}$ is __________.

f) Did all of your choices for your partial derivatives agree? Why or why not, can you determine where you made a mistake if you made one? Did slicing help you determine the signs of your partial derivatives?
Contour Plot 4:

a) From plot, I think $f_x$ is __________, $f_y$ is __________, $f_{xx}$ is __________, $f_{yy}$ is __________, and $f_{xy}$ is __________.

b) Represent the surface with Play Doh using the contour plot. Then pick which graph you think your contour plot/Play Doh surface represents. Attach picture of surface to your document that contains all pictures.

c) From my slice, Here are my my graphs on the xz and yz axes:

\[
\begin{align*}
&z \\
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**Contour Plot 5:**

a) From plot, I think $f_x$ is ________, $f_y$ is ________, $f_{xx}$ is ________, $f_{yy}$ is ________, and $f_{xy}$ is ________.

b) Represent the surface with Play Doh using the contour plot. Then pick which graph you think your contour plot/Play Doh surface represents. Attach picture of surface to your document that contains all pictures.

c) From my slice, Here are my my graphs on the xz and yz axes:

```
<p>| | |</p>
<table>
<thead>
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</table>
```

I think $f_x$ is ________ and $f_{xx}$ is ________. Attach picture of slice to your document that contains all pictures.

d) From my slice, I think $f_y$ is ________ and $f_{yy}$ is ________. Attach picture of slice to your document that contains all pictures.

e) Pick the equation you think represents this function and determine the signs of your partial derivatives algebraically. I think my function is ______________. From this, I found that $f_x$ is ________, $f_y$ is ________, $f_{xx}$ is ________, $f_{yy}$ is ________, and $f_{xy}$ is ________.

f) Did all of your choices for your partial derivatives agree? Why or why not, can you determine where you made a mistake if you made one? Did slicing help you determine the signs of your partial derivatives?
11.4 Ice 8—Partial Derivative Review

1. Determine the signs of $f_x$, $f_y$, $f_{xx}$, and $f_{yy}$ at the points Q, R, and P from the picture below:

<table>
<thead>
<tr>
<th></th>
<th>Q</th>
<th>R</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_x$</td>
<td>$f_x$</td>
<td>$f_x$</td>
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</tr>
<tr>
<td>$f_y$</td>
<td>$f_y$</td>
<td>$f_y$</td>
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<tr>
<td>$f_{xx}$</td>
<td>$f_{xx}$</td>
<td>$f_{xx}$</td>
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</tr>
<tr>
<td>$f_{yy}$</td>
<td>$f_{yy}$</td>
<td>$f_{yy}$</td>
<td></td>
</tr>
</tbody>
</table>

2. The level curves are shown for a function $f$. Determine whether the following partial derivatives are positive, negative, or zero at the point P.

---

2 Questions 2 and 3 are taken from Stewart’s *Calculus*
3. Determine the signs of the partial derivatives for the function at $(1, 2)$ and $(-1, 2)$:

\[
egin{align*}
&f_x(1, 2) & f_x(-1, 2) \\
&f_y(1, 2) & f_y(-1, 2) \\
&f_{xx}(1, 2) & f_{xx}(-1, 2) \\
&f_{yy}(1, 2) & f_{yy}(-1, 2)
\end{align*}
\]
12 Polar Coordinates

12.1 Defining Polar Coordinates

We are used to writing points in a two dimensional coordinate system using Cartesian coordinates: ________. Sometimes it is better to write a point using Polar Coordinates.

We pick a point called the ________ or origin and create the polar axis by drawing a horizontal ray from this point. Then we can define each point using an angle $\theta$, and a radius $r$. Thus, we can write a point as ________.

**Example 1:** Plot the points whose polar coordinates are the following: $A(3, \frac{\pi}{6})$, $B(1, 5\pi)$, $C(2, \frac{7\pi}{6})$, and $D(-2, \frac{\pi}{6})$.

Converting Between Cartesian and Polar Coordinates:

Polar to Cartesian $(r, \theta) \rightarrow (x, y)$: $x = \,$ $y = \,$

Cartesian to Polar $(x, y) \rightarrow (r, \theta)$: $r^2 = \,$ $\tan \theta = \,$

**Example 2:** a) Express the point with polar coordinates $P(-2, \pi)$ in Cartesian coordinates.

b) Express the point with polar coordinates $P(3, \frac{3\pi}{4})$ in Cartesian coordinates.
Example 3: Express the point with Cartesian coordinates $Q(3, 3)$ in Polar coordinates.

Voting Question 1: Which of the following are polar coordinates for the Cartesian point $Q(-1, \sqrt{3})$?

a) $(2, \frac{5\pi}{6})$

b) $(-2, \frac{5\pi}{6})$

c) $(-2, \frac{-\pi}{3})$

d) $(-2, \frac{2\pi}{3})$

Example 4: Convert the following polar equations into Cartesian:

a) $r^2 = 5$

b) $r \cos \theta = 5$

c) $r^2 \sin(2\theta) = 1$
12.2 ICE 9 – Polar Coordinates

Voting Questions!

1. True or False: A point in the xy-plane has a unique representation in polar coordinates.
   (a) True, and I am very confident.
   (b) True, but I am not very confident.
   (c) False, but I am not very confident.
   (d) False, and I am very confident.

2. The equation $\theta = \frac{\pi}{3}$ represents what type of graph in the Cartesian plane?
   (a) horizontal line
   (b) vertical line
   (c) a line through the origin
   (d) circle

3. The following graph is which function expressed using polar coordinates?
   (a) $r = 2 \sin \theta$
   (b) $r = 1$
   (c) $r = \cos \theta$
   (d) $r = 2 \cos \theta$
12.3 Polar Curves

How could we represent a line through the pole using Polar coordinates? How about a circle centered at the pole?

Example 4: Describe the following polar curves.

a) \( r = -4 \)  
b) \( r = 4 \sin \theta \)  
c) \( r = \theta \)

12.4 Graphing Polar Equations

Plotting points is a time consuming way to graph polar equations. Often it is better to use the following:

**Procedure: Cartesian to Polar Method for Graphing** \( r = f(\theta) \):

**Step 1:** Graph \( r = f(\theta) \) as if \( r \) and \( \theta \) were Cartesian coordinates with \( \theta \) on the x-axis and \( r \) on the y-axis.

**Step 2:** Use the Cartesian graph in Step 1 as a guide to sketch the points \((r, \theta)\).

**Example 1:** Use the graph of \( r = f(\theta) \) on the left to sketch the polar equation.
Example 2: Graph $r = 1 - \sin \theta$.

Example 3: Graph $r = 3 \cos(2\theta)$.
12.4.1 Circles In Polar Coordinates

The equation \( r = a \) describes a circle of radius \( a \) centered at \( (0, 0) \).

The equation \( r = 2a \sin \theta \) describes a circle of radius \( 2a \) centered at \( (0, 0) \).

The equation \( r = 2a \cos \theta \) describes a circle of radius \( 2a \) centered at \( (0, 0) \).

12.4.2 More Complex Curves In Polar Coordinates

There are more complicated polar equations you can graph. With the technology you can use today to graph these equations, it may seem like it is unnecessary to learn to graph polar curves by hand. I argue that it is good to understand the theory behind how we graph these equations because when in doubt you can always go back to the theoretical technique for complex examples.

For example suppose we wanted to graph \( r = \sec \theta - 2 \cos \theta \).

Below is the Cartesian graph \( y = \sec x - 2 \cos x \).

Thus the polar graph is:

Sometimes we have to break up our equation into two parts. For example, when graphing \( r^2 = 4 \cos(2\theta) \), we should split it into two branches: \( r = \sqrt{4 \cos(2\theta)} \) and \( r = -\sqrt{4 \cos(2\theta)} \).

Here are the graphs of \( y = \sqrt{4 \cos(2x)} \) and \( y = -\sqrt{4 \cos(2x)} \):

Why do you suppose there are gaps in this graph?

The polar equation for this “lemniscate” is
12.5  ICE 10 – Polar Graphs

1. Sketch the polar curves using the Cartesian to polar procedure to sketch the graph. You can check your sketch with your calculator or Desmos.

a) \( r = 3 \cos(\theta) \).

b) \( r = 1 + \cos(\theta) \). This is called a cardioid.
12.6 Derivatives and Tangents Lines in Polar Curves

When we find a tangent line for a polar curve $r = f(\theta)$, we think of $\theta$ as a \underline{________}. Then we can think of $x$ and $y$ in the following way:

$x = r \cos \theta \rightarrow x(\theta) =$

$y = r \sin \theta \rightarrow y(\theta) =$

Using the product rule we can find $x'(\theta) =$

Similarly $y'(\theta) =$

Then $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} =$

Horizontal tangents occur where $\frac{dy}{d\theta} =$\underline{________}, as long as $\frac{dx}{d\theta} \neq 0$.

Vertical tangents occur where $\frac{dx}{d\theta} =$\underline{________}, as long as $\frac{dy}{d\theta} \neq 0$. 
Example 4: Find the slope of the tangent line to $r = 2 - \sin \theta$ at $\theta = \frac{\pi}{3}$

### 12.7 Calculating Polar Areas

As one would expect, we can calculate the area of the region bounded by a polar equation. To derive this, we use the area of a sector of circle, $A = \frac{1}{2}r^2\theta$ and slice up our region using sectors and create a Riemann sum.

The Area of a polar region $R$ bounded by $r = f(\theta)$ between $a \leq \theta \leq b$ is

The Area of a polar region $R$ bounded by $r = f(\theta)$ and $r = g(\theta)$ is

Note: The area is not always in form of $\int_1 - \int_2$. Can you draw an example of a region you would add to find the area between curves?
Example 1: We graphed the polar curve \( r = 3 \cos(2\theta) \) in our notes last time. Set up an integral that represents the area enclosed by one of the loops. What can we use?

Example 2: In our ICE sheet, we graphed both \( r_1 = 3 \cos \theta \) and \( r_2 = 1 + \cos \theta \).

a) Find all points of intersection. Assume \( 0 \leq \theta \leq 2\pi \)

b) Now set up an integral to find the area of the region that lies outside of \( r_2 = 1 + \cos \theta \) and inside \( r_1 = 3 \cos \theta \). Recall they intersect at \( \theta = \frac{\pi}{3} \).
Neat: Recall there is no elementary indefinite integral for the Gaussian function $f(x) = e^{-x^2}$. This function shows up in many applications including statistics and quantum mechanics. In Calculus II, you may have used numerical methods to approximate the Gaussian Function over finite intervals. You can actually use polar coordinates a little real analysis to show that $\int e^{-x^2} \, dx = \sqrt{\pi}$.

12.8 Arc Length

Recall when we found the first derivatives of polar equations, we thought of $\theta$ as a parameter. Then using the product rule, we found that
\[
\frac{dx}{d\theta} = x'(\theta) = \quad \text{and} \quad \frac{dy}{d\theta} = y'(\theta) =
\]

Now let’s remember how we found the arc length for a parametric curve $x = f(t)$ and $y = g(t)$ for $\alpha \leq t \leq \beta$ where $f'$ and $g'$ are continuous and our curve is transversed once. We found our Arc Length =

So for $x = f(\theta)$, $y = g(\theta)$ for $a \leq \theta \leq b$, we would have our Arc Length =

We can simplify $(\frac{dy}{d\theta})^2 + (\frac{dx}{d\theta})^2 = \quad \text{using} \cos^2 \theta + \sin^2 \theta = 1$, and get a more simplified version of the arc length for a polar curve.

Thus the Arc Length of a polar curve $r = f(\theta)$ for $a \leq \theta \leq b$ is

Example 3: Sketch the graph of and set up an integral to find the length of the curve $r = e^\theta$ for $0 \leq \theta \leq 2\pi$. 
12.9  ICE 11 – Polar Calculus

1. Find the points on \( r = e^{\sqrt{3}\theta} \) where the tangent line is vertical.

2. In our notes 12.7, we graphed both \( r = 3 \cos \theta \) and \( r = 1 + \cos \theta \) and found all points of intersection of these curves. We also found the area inside the circle and outside \( r = 1 + \cos \theta \). Now set up an integral that find the area the region that lies in both curves. Use your calculator to compute this integral.
13 Multivariate Limits and Continuity

The definition of a limit for multivariate functions is pretty much the same for single variable functions.

**Definition:** \( \lim_{(x,y) \to (a,b)} f(x,y) = L \) if \( |f(x,y) - L| \) can be made small if \((x,y)\) is sufficiently close to \((a,b)\). But now we have lots of paths...

Consider the paths to the origin...

![Path Diagram](image)

**Key Idea:** We need to take our limit along all paths towards \((a,b)\). If we can find two paths that both approach \((a,b)\), but that go to different limits, then our limit _______ _______ _______ _______!

**Important:** We can only follow paths that are in the ____________ of our function and that go to the limit point!

**Remember:** A limit can exist even if the function doesn’t exist at a point.

13.1 Continuity of Multivariate Functions

**Definition** A function \( f(x,y) \) is continuous at \((a,b)\) if \( f \) is defined at \((a,b)\).

**Theorem:** If \( f(x,y) \) is a polynomial or a rational function \( \frac{p(x,y)}{q(x,y)} \) where \( q(x,y) \neq 0 \), then \( f \) is continuous.

**Theorem:** If \( g(x,y) \) is continuous at \((a,b)\) and \( f \) is continuous at \( g(a,b) \) (as a single variable function), then \( f \circ g \) is continuous at \((a,b)\). So in other words, \( \lim_{(x,y) \to (a,b)} f(g(x,y)) = \)

**Key Idea:** We need to be careful when we have points on the edge of domains.
13.2 Evaluating Multivariate Limits

Step 1: Check to see if the limit point is in the domain or on the boundary of the domain. If the former, we can ____________________________ otherwise, we need to try other methods.

Example 1: Determine $\lim_{(x,y) \to (e^2,4)} \ln \sqrt{xy}$.

If the point is on the boundary of the domain, here are some techniques...

Key Technique 1: Sometimes you can separate the variables to turn the multivariable limit into a single variable limit which means we can use L’Hopital.

Example 2: Determine $\lim_{(x,y) \to (0,3)} \frac{1 - \cos(x)}{xy}$.
Voting Question 1: Archer, the math cat, trying to determine \( \lim_{(x,y) \to (a,b)} f(x,y) \). He has found that this limit goes to 0 along the following paths: \( x = 0, y = 0, y = x, \) and \( y = x^2 \). He decides the limit must be 0. Is Archer’s reasoning valid? 

a) Yes!

b) No!

c) I am not sure

Key Technique 2: To show a limit does not exist, try easy paths like along the lines \______, \______, \______, or \_______. Note these paths must be in the domain and go to the limit point. For example, we can’t use the path \( x = 0 \) if we are approaching \((0,2)\).

Example 3: Determine \( \lim_{(x,y) \to (0,0)} \frac{4xy}{2x^2 + y^2} \).
**Key Technique 3:** Sometimes limits at a point like \((0,0)\) may be easier to evaluate by converting to polar coordinates. Remember that the same limit must be obtained as \(r \to 0\) along all paths to \((0,0)\).

**Example 4:** Determine \(\lim_{(x,y) \to (0,0)} \frac{\sin(x^2 + y^2)}{5x^2 + 5y^2}\).

**Example 5:** Determine \(\lim_{(x,y) \to (0,0)} \frac{x^2}{x^2 + y^2}\).
14 Linear Approximations of the Derivative

14.1 Tangent Plane Revisited

**Definition:** Consider $f(x, y) = z$ where $f$ is differentiable at $(a, b)$ & $\nabla f(a, b) \neq \mathbf{0}$. Then the equation of the tangent line at the point $(a, b)$ is given by:

$$z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b).$$

**Definition:** Consider $f(x, y, z) = 0$ where $f(x, y, z)$ is differentiable at $(a, b, c) & \nabla f(a, b, c) \neq \mathbf{0}$. Then the equation of the tangent line at the point $(a, b, c)$ is given by:

$$z = f(a, b, c) + f_x(a, b, c)(x-a) + f_y(a, b, c)(y-b) + f_z(a, b, c)(z-c).$$

**Example 1:** Find the equation of the tangent plane to the surface $f(x, y) = x^2y + y^2$ at $(3, 4)$.

Voting Question 1: Suppose that $f(x, y) = 2x^2y$. What is the tangent plane to this function at $x = 2$, $y = 3$?

(a) $z = 4xy(x - 2) + 2x^2(y - 3) + 24$
(b) $z = 4x(x - 2) + 2(y - 3) + 24$
(c) $z = 8(x - 2) + 2(y - 3) + 24$
(d) $z = 24(x - 2) + 8(y - 3) + 24$
(e) I want to go home. =)

Voting Question 2: The figure below shows the contour map of the function $f(x, y)$. The tangent plane approximation to $f(x, y)$ at the point $P = (x_0, y_0)$ is $f(x, y) \approx c + m(x - x_0) + n(y - y_0)$. What are the signs of $c$, $m$, & $n$?

(a) $c > 0, m > 0, n > 0$
(b) $c < 0, m > 0, n < 0$
(c) $c > 0, m < 0, n > 0$
(d) $c < 0, m < 0, n < 0$
(e) $c > 0, m > 0, n < 0$
14.2 The Differential

In Calculus I, we introduced the concept of the differential. Recall, the differential of \( y = f(x) \) is \( dy = f'(x)dx \). \( dy \) represents the amount that the ________ _________ rises or falls when \( x \) changes by amount \( dx = \Delta x \). The change in \( y \), \( \Delta y \), represents the amount the ________ rises or falls when \( x \) changes by amount \( dx \).

Why do we care? \( dy \) is a good approximation of ________ and can be much easier to find.

We now have the analogous situation in Calc III, only now we have a tangent plane and not a tangent line.

**Definition:** Given a differentiable function \( z = f(x,y) \), let \( dx \) and \( dy \) be the differentials of \( x \) and \( y \) respectively, then the differential of \( f \), denoted as \( df(x,y) \), \( df \), or \( dz \) is

Sometimes this is called the differential of \( z \) or the differential of the dependent variable.

**Example 2:** a) Consider the surface \( z = x^2 + y^2 \). Use the tangent plane to the surface at \((3, 4)\) to estimate \( f(2.9, 4.2) \). What are \( \Delta z \) and \( dz \) in this case?

a) Use the tangent plane to the surface at \((3, 4)\) to estimate \( f(2, 2) \). What are \( \Delta z \) and \( dz \) in this case?
15 Differentiability of Multivariate Functions

Derivatives are slopes of tangent line. What does our curve look like as we zoom in on the graph?

Even though we have been blissfully computing and approximating partial derivatives, we haven’t talked about what it means for \( f(x,y) \) to be differentiable.

**Definition:** Given \( z = f(x,y) \), the actual change in \( z \) at \((a,b)\) is \( \Delta z = \)

\[
\Delta z =
\]

**Definition:** \( z = f(x,y) \) is **differentiable** at \((a,b)\) if both \( f_x \) and \( f_y \) exist and
\[
\Delta z =
\]

where \( \epsilon_1 \) and \( \epsilon_2 \) are functions that only depend on \( \Delta x \) (the change in \( x \)) and \( \Delta y \) (the change in \( y \)) and we have that \((\epsilon_1, \epsilon_2) \to \) as \((\Delta x, \Delta y) \to \). Of course \( a \) and \( b \) are fixed.

**Definition:** \( f(x,y) \) is **differentiable on an open region** \( R \) if it is differentiable at \( \) in \( R \).

**Theorem:** If \( f_x(x,y) \) and \( f_y(x,y) \) both exist and are both \( \) on an open region \( R \) and \((a,b)\) is a point in \( R \), then \( f(x,y) \) is \( \) at \((a,b)\).

**Note:** Just because \( f_x(a,b) \) and \( f_x(a,b) \) both exist, doesn’t necessarily mean \( f \) is \( \) at \((a,b)\).

**Theorem:** If \( f \) is differentiable at \((a,b)\), then \( f \) is \( \) at \((a,b)\).

Is the converse true?

**Contrapositive Thm:** If \( f \) is NOT \( \) at \((a,b)\), then \( f \) is NOT \( \) at \((a,b)\).
Example 1: Discuss the continuity and differentiable of

\[ f(x) = \begin{cases} 
\frac{4xy}{2x^2 + y^2} & (x, y) \neq 0 \\
0 & (x, y) = 0 
\end{cases} \]

Recall in our section on limits, we found that \( \lim_{(x,y) \to (0,0)} \frac{4xy}{2x^2 + y^2} = \) ________
16 The Chain Rule

Chain Rule, Chain Rule, Chain Rule... we can’t get enough of you! You are the best and most useful thing I know! -Anonymous Calculus Student

**Chain Rule Version 1:** Suppose $z$ is a differentiable function of $x$ & $y$. $x$ and $y$ are differentiable functions of $t$, then $\frac{dz}{dt} =$

**Example 1:** a) Given $z = \sqrt{x^2 + y^2}$ where $x = \cos(3t)$ & $y = \sin(3t)$, determine $\frac{dz}{dt}$.

b) Find $\frac{dz}{dt}$ an alternative way.
Chain Rule Version 2: Suppose $z$ is a differentiable function of $x$ and $y$ and both $x$ and $y$ are function of $s$ and $t$. (So in other words, we have $z = \underline{\text{function of } s, t}$ and $x = \underline{\text{function of } s, t}$ and $y = \underline{\text{function of } s, t}$.) Then $z = f(x(s, t), y(s, t))$ has the following first partial derivatives:

$z_s =$  

$z_t =$  

Example 2: Example Let $z = f(x, y) = e^{x+y}$ and let $x = st$ and $y = s - t$. What is the partial derivative of $f$ with respect to $s$? With respect to $t$?

Voting Question 1: Suppose $R = R(u, v, w)$, $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$. In the chain rule, how many terms will you have to add up to find the partial derivative of $R$ with respect to $x$?

(a) 1  
(b) 2  
(c) 3  
(d) 4  
(e) 5
Example 3: Suppose the radius of a right circular cone is decreasing at a rate of 2 in/s, while its height is increasing at a rate of 3 in/s. At what rate is the volume of the cone changing when the radius is 100 in. and the height is 120 in. (The volume formula for a right circular cone is \( V = \frac{1}{3} \pi r^2 h \).)

Voting Question 2: A company sells regular widgets for $4 apiece and premium widgets for $6 apiece. If the demand for regular widgets is growing at a rate of 200 widgets per year, while the demand for premium widgets is dropping at the rate of 80 per year, the companys revenue from widget sales is:

(a) staying constant  
(b) increasing  
(c) decreasing  
(d) we cannot tell from this information

16.1 Implicit Differentiation

In Calculus I, you learned how to differentiate a function of \( x \) and \( y \). What was that process?
Example: Find \( \frac{dy}{dx} \) for \( x^3 + 4xy^2 = y^3 \)
Now consider if we think of \( x^3 + 4xy^2 = y^3 \) as \( f(x, y) = 0 \). In other words, we have \( f(x, y) \) differentiating both sides with respect to \( x \) using the chain rule like we did in calc I gives us 

\[
\frac{dy}{dx} = \]

**The Multivariate Way To Implicit Differentiation:**

1) Set the equation/function equal to 0.
2) Find \( f_x \).
3) Find \( f_y \).
4) Then \( \frac{dy}{dx} = -\frac{f_x}{f_y} \).

**Example 4:** Find \( \frac{dy}{dx} \) for \( x^3 + 4xy^2 = y^3 \) using your Calc III method.

**Extension to 3 variables:** Suppose \( f(x, y, z) = 0 \). Then using the chain rule and differentiating both sides with respect to \( x \) allows us to see that:

\[
\frac{\partial z}{\partial x} = \text{______________} \quad \text{and} \quad \frac{\partial z}{\partial y} = \text{______________}
\]
16.2 ICE 12– Chain Rule

1. The figures below show contours of $z = z(x, y)$, $x$ as a function of $t$, and $y$ as a function of $t$. Decide if $\frac{dz}{dt}|_{t=2}$ is

(a) Positive
(b) Negative
(c) Approximately zero
(d) Cant tell without further information

2. Use a tree diagram to write out the chain rule for the case where $w = f(t, u, v)$, $t = t(p, q, r, s)$, $u = u(p, q, r, s)$ and $v = v(p, q, r, s)$ are all differentiable functions.

3. Consider $ye^{-x} + z \cos(x) = 0$

   a) Use the extension version of the chain rule to determine $\frac{\partial z}{\partial x}$.

   b) Determine $\frac{\partial x}{\partial z}$. 

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17 Directional Derivatives and Gradient

We love partial derivatives $f_x$ and $f_y$, but although they tell us a lot about our function, they don’t tell us everything. Suppose I am standing on the surface below.

- What tells me the rate of change/slope of the surface in the direction parallel to the $x$ axis?
- What tells me the rate of change/slope of the surface in the direction parallel to the $y$ axis?
- What tells me the rate of change/slope of the surface in a direction other than a coordinate direction (like say northwest)?
- If I drop a ball at my feet on this surface and let it roll, in which direction will it roll?
- If I am hiking and I want to follow the steepest path, where do I step?

These latter questions will be answered in this section. (Get excited!)

**Review:** Recall, the direction of any vector $\mathbf{v}$ is determined by the unit vector $\mathbf{u} = \ldots$

**Definition 1:** Let $\mathbf{u} = <a, b>$ be a unit vector, then the directional derivative of $f$ at $(x_0, y_0)$ in the direction of $\mathbf{u}$ is $D_{\mathbf{u}} f(x_0, y_0) = \ldots$

Provided the limit exists!

**Comment:** If $f$ is a differentiable function of $x$ and $y$, then the Directional Derivative __________.

**Why do we care?** The directional derivative allows us the find the rate of change of $f$ in an __________ direction!

**Example 1:** a) Given $f(x, y) = e^{2xy}$, determine the directional derivative in the direction of $\mathbf{i}$ at $(x_0, y_0)$.
b) Given \( f(x, y) = e^{2xy} \), determine the directional derivative in the direction of \( \mathbf{j} \) at \((x_0, y_0)\).

Our discovery in Example 1, allows us to define the directional derivative in terms of the __________.

It turns out we have a special name for \( \langle f_x, f_y \rangle \)...  

17.1 Gradient

Definition: Suppose \( f \) is differentiable at \((a, b, c)\), the gradient of \( f \) at \((a, b, c)\) is the vector valued function \( \nabla f(a, b, c) = \)

Recall we can denote \( z = f(x, y) \) as \( f(x, y, z) = 0 \)

\textbf{Beware:} \( f'(x) \) is a __________, \( \nabla f \) is a __________

Example 2: Consider the surface \( f(x, y) = x^2 y + y^2 \).

a) Is \( f \) differentiable everywhere?  
b) Find \( \nabla f \) and \( \nabla f(3, 4) \).

\textbf{Properties of} \( \nabla f \):

1) \( \nabla(f + g) = \)
2) \( \nabla(af) = \)
1) \( \nabla(fg) = \)
17.2 Directional Derivatives Definition

**Definition 2:** Suppose $f$ is a differentiable function of $x$ & $y$, then the directional derivative in the direction of unit vector $\mathbf{u} = <a, b>$ is $D_{\mathbf{u}} f(x, y) =$

That is, $D_{\mathbf{u}} f(x, y) =$

**Example 3:** Given $f(x, y) = e^y \sin(x)$, what is the directional derivative of $f$ at $(\frac{\pi}{6}, 0)$ in the direction of $<\sqrt{3}, 1>$?

---

**Voting Question 1:** In which direction is the directional derivative of $z = x^2 + y^2$ at the point $(2, 3)$ most positive?
(a) $\mathbf{i}$
(b) $\mathbf{i} - \mathbf{j}$
(c) $-\mathbf{i} + \mathbf{j}$
(d) $\mathbf{i} + \mathbf{j}$

Recall, $D_{\mathbf{u}} f(x, y) = \nabla f \cdot \mathbf{u} =$

- At what angle will our directional derivative at $(x_0, y_0)$ have its maximum value?
- When will the directional derivative have its minimum value?
- When will the direction derivative be 0?

**Summary:** Suppose $f$ is a differentiable function at $(x_0, y_0)$

- $f$ has its maximum rate of increase in the direction of ________ and this rate of increase is

- $f$ has its maximum rate of decrease in the direction of ________ and this rate of decrease is

- The directional derivative is 0 in any direction that is ________ to $\nabla f$
Recall level curves of $z = f(x, y)$ are projections onto planes parallel to the $xy$ plane.

**Idea:** Since the value of $f$ is constant along each level curve, its rate of change is _______. Which means our directional derivative is _______. Thus $0 = D_u f(x_0, y_0) = \nabla f(x_0, y_0) \cdot u_0$. Thus the gradient is _______ to the level curves. In other words, it is _______ to the line tangent to the level curve.

**Example 4:** Find a unit vector in the direction in which $f(x, y) = e^{2x+y}$ increases most rapidly at $(0, 1)$.  

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Example 5: Let \( f(x, y) = xy \).

a) What is the maximum rate of change at (0, 0).

b) Is the function identically zero near the origin? Can we talk about the direction of maximal change at (0, 0)?

Warning: If \( \nabla f(x_0, y_0) = \text{________} \), then we can’t determine the direction of maximal change at \((x_0, y_0)\).

Voting Question 2: The surface of a hill is modeled by \( z = 25 - 2x^2 - 4y^2 \). When a hiker reaches the point (1, 1, 19), it begins to rain. She decides to descend the hill by the most rapid way. Which of the following vectors points in the direction in which she starts her descent?

(a) \( <-4x, -8y> \)
(b) \( <4x, 8y> \)
(c) \( <4x, -8y> \)
(d) \( <-4x, 8y> \)
(e) None of the above

Voting Question 6: Which of the vectors show on the contour diagram of \( f(x, y) \) in the figure below could be \( \nabla f \) at the point at which the tail is attached. (Pick one.)

(a) A  
(b) B  
(c) C  
(d) D
18 Optimization

18.1 Max and Mins

Again, this section confirms that Calc III is in the same spirit of Calc I in its definitions of max and mins...

**Definition:** Suppose \( f \) is a continuous function on its domain \( D \). Let \( p_0 \) be a point in \( D \). Then we say \( f(p_0) \) is a

- Global Max of \( f \) on \( D \)
- Global Min of \( f \) on \( D \)
- Local Max value
- Local Min value

**Note:** If \( f(p_0) \) is either a global max or min, we say it is a *global extreme value* and if \( f(p_0) \) is either a local max or min, we say it is a *local extreme value*.

Remember how in Calc I, if we have a continuous function on a closed and bounded interval, our function reaches a _______ max and a _______ min on that interval? Of course you do! Now we have the same idea for Calc III...

**Theorem:** If \( f \) is continuous on a _______ and _______ set \( S \), then \( f \) attains both a _______ _______ and _______ _______ on \( S \).

18.2 Critical Points

In Calc I, our max and mins were attained at _______ points or _______ points.

**Definition:** An interior point, \((a, b)\) is a *critical point* if \((a, b)\) is in the domain of \( f(x, y) \) and either one of the following is true:

In Calc III, our Critical Points are of 3 types:

1. **Boundary Points:**
2. **Stationary Points:** an interior point at which \( f \) is differentiable and the \( \nabla f = \) _______.
3. **Singular Points:** an interior point at which \( f \) is _______ differentiable at.

**Theorem:** Extreme values of \( f(x, y) \) on a set \( S \) are also _______ points.
Example 1: Find the critical points of \( f(x, y) = x^4 + 2y^2 - 4xy \).

Example 2: Find the critical points of \( f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - 4x - 9y \).

Voting Question 1: Which of these functions has a critical point at the origin?
(a) \( f(x, y) = x \cos y \)
(b) \( f(x, y) = x^2y + 4xy + 4y \)
(c) \( f(x, y) = x^2 + 2y^3 \)
(d) \( f(x, y) = x^2y^3x^4 + 2y \)
(e) All of the above

Second Derivative Test: Suppose the 2\(^{nd}\) partial derivatives of \( f \) are continuous for all points in an open disk at \((a, b)\) where \( f_x(a, b) = f_y(a, b) = 0 \) and let \( D = D(a, b) = \)

Then if
- \( D(a, b) > 0 \) AND \( f_{xx}(a, b) < 0 \), then \((a, b)\) is a _______ ________
- \( D(a, b) > 0 \) AND \( f_{xx}(a, b) > 0 \), then \((a, b)\) is a _______ ________
- \( D(a, b) < 0 \), then \((a, b)\) is a _______ ________
- \( D = 0 \), then
Note: \((a, b)\) is a saddle pt if we can find values such that \(f(x, y) > \_\_ \& f(x, y) < \_\_.\)

Example 3: Classify the critical points of \(f(x, y) = x^4 + 2y^2 - 4xy\). Recall in Example 1 we found our CP’s to be \((0,0),(1,1) & (-1,-1)\) and \(f_x = 4x^3 - 4y\) and \(f_y = 4y - 4x\).

Voting Question 2: How would you classify the function \(f(x, y) = x^4 - y^4\) at the origin?
(a) This is a local maximum.
(b) This is a local minimum.
(c) This is a saddle point.
(d) We cannot tell.
(e) This is not a critical point.

Fact: If \(f \geq 0\), \(f^n\) has the same \_\_\_\_\_\_\_\_\_ and \_\_\_\_\_\_\_\_\_ as \(f\) for any \(n\).
Helpful hint: Sometimes it is easier to optimize \(f^n\) rather than \(f\).

Example 4: Find the point on the surface \(z^2 = x^2 + xy + y + 1\) closest to the origin.
18.3 Using the Contour Map to Classify Critical Points

Classifying Critical Points using a Contour Map:

- Local Maxima: at the _________ of the contour and all contour _________ as we move towards the point.

- Local Minima: at the _________ of the contour and all contour _________ as we move towards the point.

- Saddle Point: Usually at the intersection of 2 contour lines. We increase in one direction and decrease when we move in “basically a perpendicular direction” (not necessarily exactly perpendicular). Idea: Go up one way and down in another direction.

Voting Question 3: Use the contour grid to determine which of the following points are critical points.
(a) A and C
(b) A, C, and D
(c) A, B, and C
(d) A, B, C, and D
18.4 ICE 13– Optimization

Follow the steps below to find the absolute max and min values of \( f(x, y) = x^2 + y^2 + x^2 y + 4 \) on the region \( D = \{(x, y) : |x| \leq 1, |y| \leq 1\} \).

1. Sketch the region \( D \).

2. Find the critical points of \( f \), and then find the values of \( f \) at the critical points that are inside \( D \).

3. On the top part of the boundary of \( D \), \( f \) reduces to a function of one variable on a closed interval. Use techniques from Calculus I to find the extreme values of \( f \) on the top part of the boundary of \( D \).
4. Repeat the previous question for the bottom part of the boundary of \( D \).

5. Repeat the previous question for the right part of the boundary of \( D \).

6. Repeat the previous question for the left part of the boundary of \( D \).

7. The largest of the values from questions 2-6 is the absolute max and the smallest of these values is the absolute min. Find the absolute max and absolute min on \( D \).
19 Lagrange Multipliers

19.1 Optimizing with Constraints

Goal: To find a max or min value of $f(x, y)$ subject to a constraint function $g(x, y) = c$.

We want to find the largest and smallest values of $c$ such that the level curve $f(x, y) = k$ are tangent to ________.

When does this occur? 

![Diagram of Lagrange Multipliers]

**Theorem:** Suppose $f(x, y)$ is a differentiable function in a region that contains the smooth curve given by $g(x, y) = 0$. If $f$ has a local extreme value on the curve $g(x, y)$ at $(a, b)$, then $\nabla f(a, b)$ is ________ to the line tangent to $g(x, y)$ at $(a, b)$. Furthermore, as long as $\nabla g(a, b) \neq 0$, there exists a real number $\lambda$ such that $\nabla f(a, b) = \lambda \nabla g(a, b)$.

**Method of Lagrange Multipliers:**
Suppose $f$ and $g$ are differentiable on $\mathbb{R}^2$ with $\nabla g(x, y) \neq 0$ and $g(x, y) = 0$.
To find the max or min values of $f(x, y)$ subject to $g(x, y) = 0$:

**Step 1:** Find $x, y, \& \lambda$ such that ________ = ________ AND ________ = ________.

**Step 2:** From the different points $(x, y)$ you found in Step 1, plug them into $f(x, y)$ and pick the largest value for your ________ and the smallest value for your ________.

**Key Idea:** $\nabla f = \lambda \nabla g$ at points in which the $\nabla f$ is parallel to $\nabla g$ which is when both $\nabla f$ and $\nabla g$ are orthogonal to the tangent line to the constraint function.

---

Voting Question 1: For $0 \leq x \leq 5$, find the maximum and minimum values of $f$ on $g = c$.

(a) max = 5, min = 0  
(b) max = 4, min = 0  
(c) max = 3, min = 2  
(d) max = 4, min = 2

19.2 2 Constraints

Suppose we want to find extreme values of $f(x, y, z)$ and we have more than one constraint, say two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$. So we are looking for the max and min values of $f$ which lie on the intersection of $g(x, y, z) = k$ and $h(x, y, z) = c$. In this case our $\nabla f$ is determined by both $\nabla g$ and $\nabla h$. So we now have Lagrange multipliers for each constraint function. So in this example, we have

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

Therefore we now solve 5 equations:

Example 1: Set up, but do not solve, the equations you would solve to find the extreme values of $f(x, y, z) = z$ subject to $x^2 + y^2 = z^2$ and $x + y + z - 24 = 0$
Example 2: Find the maximum and minimum values of $f(x, y) = \frac{1}{x} + \frac{1}{y}$ subject to the constraint $\frac{1}{x^2} + \frac{1}{y^2} = 1$. 
**Example 3:** Find the minimum value of $f(x, y) = x^2 + 4xy + y^2$ subject to the constraint $x - y = 6 - 0$. 
Example 4: Find the max and min values of $f(x, y) = xy^2$ subject to $x^2 + y^2 = 1$. 
19.3  ICE 14– Lagrange Multipliers

1. Use Lagrange Multipliers to find the extreme values of the function $f(x, y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$.

2. Below is the graph of the unit circle along with several level curves of the function $f$. Use the graph to determine the extreme values of the function $f(x, y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$. 

![Graph of the unit circle and level curves](image-url)
20 Multivariate Integration

20.1 Double Integrals over Rectangular Regions

When we were in Calculus I, we derived our definite integral by using a Riemann sum. In this case, we partitioned our interval \([a, b]\) and summed up the areas of rectangles. We have the same idea in Calculus III, only now instead of an integral, we have a rectangle, and we consider a small change in the area, \(\Delta A\). When evaluating the limit of a Riemann sum of \(f(x, y)\) over a partitioned set rectangles, we find the volume of the solid bounded by \(f(x, y)\) over the rectangular region.

\[\text{Theorem: If } f \text{ is bounded on a closed rectangle } R \text{ and if it is continuous there except on a finite number of smooth curves, then } f \text{ is integrable on } R. \text{ In particular, if } f \text{ is continuous on all of } R, f \text{ is integrable.}\]

\[\text{Definition: If } f \text{ is an integrable function, the volume of the solid bounded by } z = f(x, y) \text{ over the regional } R \text{ is called the double integral of } f \text{ over } R \text{ and is denoted:}\]

While evaluating double integrals over a rectangular region, we have similar results as with single integrals:

- It is linear: \(\iint_R (k \cdot f(x, y) + g(x, y))dA =\)
- Additive on non-overlapping rectangles: \(\iint_R f(x, y)dA =\)
- If \(f(x, y) \leq g(x, y)\), then \(\iint_R f(x, y)dA \leq\)

Example 1: Evaluate \(\iint_R f(x, y)dA\) for the function below

\[f(x, y) = \begin{cases} 2, & 0 \leq x \leq 5, 0 \leq y \leq 1 \\ 3, & 0 \leq x \leq 5, 1 \leq y \leq 3 \end{cases}\]
**Voting Question 1:** Suppose the contour plot shown shows the height of a pile of dirt in feet. Which is a sum that is using the midpoint approximation rule to approximate the volume of the dirt pile?

(a) 0*1 + 0*1 + 6*1 + 6*1  
(b) 9*1 + 9*1 + 9*1 + 9*1  
(c) 1*5 + 1*5 + 1*9 + 1*9  
(d) 6*1 + 6*1 + 9*1 + 9*1

Notice in Example 1, we could have written our integral in the following way:

\[ f(x, y) = \begin{cases} 
2, & 0 \leq x \leq 5, 0 \leq y \leq 1 \\
3, & 0 \leq x \leq 5, 1 \leq y \leq 3 
\end{cases} \]

This form of the integral is called an ________ integral.

**FUBINI’s Theorem:** Let \( f(x, y) \) be continuous on a rectangular region \( R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\} \), then \( \iint_{R} f(x, y) \, dA = \)

**Warning:** This is ONLY true for regions that are ________!

**Example 2:** Below we have the graph of the solid whose volume is given by the double integral over the rectangle \( R = \{(x, y) : 0 \leq x \leq 2, -1 \leq y \leq 1\} \). Find the volume using an iterated integral. \( \iint_{R} (x^2 + xy) \, dA \) and check that Fubini’s Theorem holds.
20.2 Iterated Integrals over General Regions

20.2.1 General (non-rectangular) Regions of Integration

Before we discussed integrating over rectangular regions which is nice because we can use _________!!! Unfortunately we often have more general regions

\[ y \text{-simple set:} \]
\[ x \text{-simple set:} \]
\[ \text{Neither:} \]
\[ \text{“Both”:} \]

When in trouble, we can cut up a set so that we have a finite number of simple sets.

**Key Idea:** When setting up our iterated integral, make sure we have the _______ bounds on the outside.

20.2.2 Iterated Integrals

**Theorem:** Suppose \( R \) is bounded by 2 continuous functions of \( x \) (say \( f(x) \leq y \leq g(x) \)) and we have \( a \leq x \leq b \), then
\[
\int \int_R f(x, y) dA =
\]

If \( R \) is bounded by 2 continuous functions of \( y \) (say \( h(y) \leq x \leq l(y) \)) and we have \( c \leq y \leq d \), then
\[
\int \int_R f(x, y) dA =
\]

**Example 1:** Evaluate the volume of the solid made by \( f(x, y) = x + y \) over the region in the xy-plane bounded by \( y = \sqrt{x} \) and \( y = x^2 \).
Example 2: Change the order of integration of \( \int_0^2 \int_{x^2}^{2x} f(x,y) dy dx \).

Example 3: Why must we change the order of integration in order to evaluate \( \int_0^1 \int_y^1 e^{x^2} dx dy \)?

Example 4: a) Sketch the tetrahedron bounded by the coordinate planes and the plane \( 3x + 4y + z - 12 = 0 \).

b) Set up an integral that represents the volume of the solid described in part a.
20.3  ICE 15– Iterated Integrals

For each of the following iterated integrals, sketch the region of integration and rewrite the integral as an iterated integral with the order of integration reversed. DO NOT perform the integrations.

2 sides!

1. \[ \int_0^1 \int_0^x xy^2 \, dy \, dx \]

2. \[ \int_0^1 \int_0^y xy^2 \, dx \, dy \]

3. \[ \int_1^2 \int_0^{2x} xy^2 \, dy \, dx \]
4. Write an integral for the volume under $f(x, y) = xy^2$ over the region of the unit circle in the $xy$–plane.
20.4 Integration over Polar Regions

Recall, we have the following conversions from Cartesian coordinates to and from polar coordinates:

\[ x = \quad \quad r^2 = \]

\[ y = \quad \quad \theta = \]

**Theorem:** Let \( f \) be continuous on the region given by \( R = \{(r, \theta) : 0 \leq a \leq r \leq b, \alpha \leq \theta \leq \beta\} \) (note we need \( \beta - \alpha \leq 2\pi \)). Then \( \iint_{R} f(r, \theta) dA = \)

**Note:** This double integral gives us an _________ not a ________!

**Example 1:** Evaluate \( \iint_{R} 4xy dA \) for \( R = \{(r, \theta) : 1 \leq r \leq 3, 0 \leq \theta \leq \frac{\pi}{2}\} \).
Example 2: Sketch the region whose area is given by the integral \[ \int_{\frac{3\pi}{4}}^{\frac{\pi}{4}} \int_{3}^{1} r dr d\theta. \]

Example 3: Determine for each of the following regions, whether we should use polar coordinates or Cartesian coordinates when integrating a function of two variables over the regions.

Example 4: Evaluate the integral \[ \iint_{R} \sqrt{x^2 + y^2} \, dA \] where \( R = \{(x, y) : 1 \leq x^2 + y^2 \leq 9, y \geq 0\} \) by first changing to polar coordinates.
21 Applications of Multivariate Integration

21.1 Density and Mass (Optional Topic)

Suppose we have a flat sheet that is soooo thin that we can treat it like it is a 2-dimenional object. Such a sheet is called a __________. You may remember this from Calc II or physics. We can use single integrals to compute the centroid or center of mass of a lamina. But we could only do this if the lamina had constant __________ or __________ that only depended on x. Now since we have double integrals, we can deal with densities that vary with respect to BOTH x and y.

**Definition:** Suppose that a lamina covers a region R in the xy-plane and has a density, $\delta(x, y)$, then the total mass of the lamina is $m =$

**Example 1:** Find the mass of the lamina bounded by $y = e^x$, $y = 0$, $x = 0$, and $x = 1$ with density $\delta(x, y) = x + y$. 

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When we have a constant density of \( K \), we defined the moment of a particle about an axis as the product of its mass and its directed distance from the axis. Thus the moment or total moments about the \( y \)-axis: \( M_y = \)

the moment or total moments about the \( x \)-axis: \( M_x = \)

We used \( M_y \) and \( M_x \) to find the center of mass or ________ point.

**Definition:** Suppose that a lamina covers a region \( R \) in the \( xy \)-plane and has a density, \( \delta(x, y) \), then the total moments about the \( y \) axis is \( M_y = \)

and the total moments about the \( x \) axis is \( M_x = \)

Thus the coordinates of the center of mass are \((\bar{x}, \bar{y})\),
where \( \bar{x} = \) and \( \bar{y} = \)

The significance of the center of mass, is that the lamina behaves as if its entire mass is concentrated at its ________ ________ _________. AKA we can balance the lamina by supporting it at its center of mass.

What if we had a constant density? Then we should have something that coincides with the single variable version...

**Example 2:** Find the total moment about the \( y \) axis of the homogeneous lamina with density \( k \) which covers the \( y \)-simple region: \( R = \{(x,y)|a \leq x \leq b, g(x) \leq y \leq f(x)\} \).
Example 3: Find the center of mass of the lamina bounded by $y = 0, y = \sqrt{4 - x^2}$ with density $\delta(x, y) = y$.
21.2 Moments of Inertia/ Rotational Inertia (Optional Topic)

**Definition:** The *moment of inertia* (sometimes called the second moment) of a particle of mass $m$ about an axis is defined to be $mr^2$, where $r$ is the distance from the particle to the __________.

Idea: In physics the kinetic energy $KE$ of our particle moving in a straight line with velocity $v$ is: $KE =$

If our particle is rotating about an axis with angular velocity $w$, its linear velocity is $v = rw$ where $r$ is as defined above -the radius of the circular path. Thus $KE =$.

The $r^2m$ portion of this is called the moment of inertia.

**Definition:** For a lamina with density $\delta(x,y)$ over region $R$, the *moment of inertia of the lamina about the x-axis* is

$I_x =$

and *the moment of inertia of the lamina about the y-axis* is

$I_y =$

and *the moment of inertia of the lamina about the z-axis* sometimes called the moment of inertia about the origin or the polar moment of inertia is

$I_z =$

**Example 4:** Find the moments of inertia of the lamina about the x,y, and z axes for a homogeneous disk centered at the origin with radius $a$ and with density $\delta(x,y) = k$. 

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Remember we can find the mass of a homogeneous disk by \( m = \) \( \times \) \( \). Thus in our Example 4, we have that \( I_z = \) \( \)

**Definition:** The *radius of gyration of a lamina about an axis* is the number \( R \) such that \( mR^2 = \) \( \) \( \) \( \) where \( \) \( \) \( \) \( \) is the mass of the lamina and \( \) \( \) \( \) is the moment of inertia about the given axis.

(So if \( m\hat{y}^2 = I_x \) and \( m\hat{x}^2 = I_y \), the point \((\hat{x}, \hat{y})\) is the point at which the mass of the lamina can be concentrated without changing the moments of inertia with respect to the coordinate axes.)

**Example 5:** Find the radius of gyration about the \( x \)-axis from Example 4.
Find the center of mass of the lamina bounded by $r = 1, r = 2, 0 \leq \theta \leq \frac{\pi}{2}$ with density $\delta(r, \theta) = \frac{1}{r}$. Hint: use symmetry to find $M_y$. 
21.4 Surface Area

In Calc II should have learned about finding the surface area of a revolution. We now develop a formula for the area of a surface defined by \( z = f(x, y) \) over a region.

**Idea:** Partition our surface. These will be curved but we use the tangent plane to create a parallelogram. Then we find the areas of our parallelograms. Recall, the area of a parallelogram with adjacent sides \( a \) and \( b \) is

**Definition:** Given a smooth parametric surface \( S \) given by the equation \( \mathbf{r}(u, v) = <x(u, v), y(u, v), z(u, v)> \) (where \( S \) is covered just one as the parameters range through the domain of \( S \) ), then the surface area of \( S \) is given by \( A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA \) \quad \text{Note} \ r_u = <x_u, y_u, z_u> \text{ and } r_v = <x_v, y_v, z_v>.

**Explicit Case:** To find the Area of a surface represented explicitly by \( z = g(x, y) \), we evaluate

\[
A(S) = \iint_R \sqrt{1 + g_x^2 + g_y^2} \, dA
\]

**Note:** Often setting up a surface area problem is easy, but often it is difficult or impossible to evaluate these integrals. So we must use a computing system or approximate the integral.

**Example 1:** a) Use the following 3 views of the surface to determine a good approximation of the portion of the graph over our square. Find the surface area of the portion of the graph of \( f(x, y) = 10 - \frac{y^2}{2} \) lying over the square with vertices \((0, 0), (4, 0), (4, 4), \& (0, 4)\).
b) Find the surface area of the portion of the graph of \( f(x, y) = 10 - \frac{y^2}{2} \) lying over the square with vertices \((0, 0), (4, 0), (4, 4), \) & \((0, 4)\).

**Example 2:** Find the surface area of the part of the paraboloid \( x = y^2 + z^2 \) that lies inside the cylinder \( y^2 + z^2 = 9 \).
21.5 ICE 17– Surface Area

1. Set up an integral to find the surface area of the part of \( z = 15 - x^2 - y^2 \) above the plane \( z = 14 \).

2. Set up an integral to find the part of \( z = 9 - x^2 \) above the xy plane with \( 0 \leq y \leq 20 \).
22 Triple Integrals

“Triple Integrals will make you smile. Triple Integrals, they last a while. Triple Integrals will help you, mister to punch tough volumes right in the kisser. Triple Integrals!”

22.1 Triple Integrals in Cartesian Coordinates

Triple integrals are “basically” double integrals with an extra variable. Whereas \( \iint_R 1\,dA = \) __________

We now have \( \iiint_D 1\,dV = \) __________

Now let \( f(x,y,z) \) be a continuous function defined on a region \( R = \{(x,y,z) : a \leq x \leq b, g(x) \leq y \leq h(x), G(x,y) \leq z \leq H(x,y)\} \) where \( g, h, G, \& H \) are continuous. Then the triple integral of \( f \) on \( R \) is evaluated as the following iterated integral:

Example 1: Find the volume of the snow cone made where the top of the cone is \( x^2 + y^2 + z^2 = 8 \) and the cone part is given by \( z = \sqrt{x^2 + y^2} \).
Example 2: Set up a triple integral for the volume of the following solid. (The tetrahedron created by the intersection of the plane $z = 1 - x - y$ and the coordinate planes.)
Example 3: Sketch the solid whose volume is given by the iterated integral \[ \int_0^1 \int_0^{1-x} \int_0^{2-2z} dydzdx. \]
Example 4: Consider the integral \[ \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} f(x, y, z) \, dz \, dy \, dx. \] Rewrite this integral as an integral of the form: \[ \int \int f(x, y, z) \, dy \, dz \, dx. \]
22.2  ICE 18– Triple Integrals

1. Set up a triple integral for the volume of the following solid, which is the piece of the unit sphere in the first octant.
22.3 Triple Integrals Using Cylindrical and Spherical Coordinates

22.3.1 Cylindrical Coordinates

Recall, when we are dealing with a 2-dimensional coordinate system, we could write points in Cartesian or Polar form. Today we will discuss two alternate coordinate systems for which we can represent coordinates in three dimensions.

So basically we have Polar Coordinates with a ________!

Converting Between Cartesian and Cylindrical Coordinates:

Cartesian to Cylindrical: \((x, y, z) \rightarrow (r, \theta, z)\):  
\[
\begin{align*}
  r &= \ldots \\
  \theta &= \ldots \\
  z &= \ldots 
\end{align*}
\]

Cylindrical to Cartesian: \((r, \theta, z) \rightarrow (x, y, z)\):  
\[
\begin{align*}
  x &= \ldots \\
  y &= \ldots \\
  z &= \ldots 
\end{align*}
\]

Voting Question 1: What are the Cartesian coordinates of the point with cylindrical coordinates \((r, \theta, z) = (4, \pi, 6)\)?

(a) \((x, y, z) = (0, -4, 4)\)
(b) \((x, y, z) = (0, 4, 6)\)
(c) \((x, y, z) = (-4, 4, 4)\)
(d) \((x, y, z) = (4, 0, 4)\)
(e) \((x, y, z) = (-4, 0, 6)\)

Voting Question 2: What are the cylindrical coordinates of the point with Cartesian coordinates \((x, y, z) = (3, 3, 7)\)?

(a) \((r, \theta, z) = (3, \pi, 7)\)
(b) \((r, \theta, z) = (3, \frac{\pi}{3}, 3)\)
(c) \((r, \theta, z) = (3\sqrt{2}, \frac{\pi}{4}, 7)\)
(d) \((r, \theta, z) = (3\sqrt{2}, \pi, 7)\)
(e) \((r, \theta, z) = (3\sqrt{2}, \pi, 3)\)
**Example 1:** Write the equation of the surface \( z = x^2 + y^2 - y \) in cylindrical coordinates.

**Voting Question 3:** The graph of \( r = 5 \) is a

(a) circle  
(b) sphere  
(c) cone  
(d) plane  
(e) cylinder

**Voting Question 4:** The graph of \( \theta = \frac{\pi}{4} \) is a

(a) circle  
(b) sphere  
(c) cone  
(d) plane  
(e) cylinder

**Example 2:** Sketch the graph of \( r^2 \cos^2 \theta + z^2 = 9 \)
22.3.2 Integration with Cylindrical Coordinates

Recall, Cylindrical Coordinates are just polar coordinates with a 

\[
\begin{align*}
\text{Cartesian} & \rightarrow \text{Cylindrical}: \\
(x, y, z) & \rightarrow (r, \theta, z): \\
r^2 & = x^2 + y^2 \\
\theta & = \arctan\left(\frac{y}{x}\right) \\
z & = z
\end{align*}
\]

\[
\begin{align*}
\text{Cylindrical} & \rightarrow \text{Cartesian}: \\
(r, \theta, z) & \rightarrow (x, y, z): \\
x & = r \cos \theta \\
y & = r \sin \theta \\
z & = z
\end{align*}
\]

Voting Question 5: Which of the following regions represents the portion of the cylinder of height 4 and radius 3 above the 3rd quadrant of the xy plane?

(a) \(0 \leq r \leq 3, 0 \leq z \leq 4, 0 \leq \theta \leq \frac{\pi}{4}\) 
(b) \(0 \leq r \leq 3, 0 \leq z \leq 4, \pi \leq \theta \leq \frac{3\pi}{2}\) 
(c) \(1 \leq r \leq 4, 0 \leq z \leq 3, \pi \leq \theta \leq \frac{3\pi}{2}\) 
(d) \(1 \leq r \leq 3, 0 \leq z \leq 4, 0 \leq \theta \leq \frac{\pi}{2}\)

Now let \(f(r, \theta, z)\) be a continuous function defined on a region \(R\) where \(R := \{(r, \theta, z) : a \leq \theta \leq b, g(\theta) \leq r \leq h(\theta), G(x, y) \leq z \leq H(x, y)\}\) where \(g, h, G, H\) are continuous. Then the triple integral of \(f\) over \(R\) in cylindrical coordinates is

\[
\int \int \int_R f(r, \theta, z) dV =
\]

Voting Question 7: Which of the following is equivalent to \(\int_{-5}^{5} \int_{0}^{3} \int_{\sqrt{25-x^2}}^{\sqrt{25-x^2}} x dudzdx\)?

a) \(\int_{0}^{3} \int_{0}^{\pi} r^2 \cos \theta d\theta dr dz\) 
(b) \(\int_{0}^{3} \int_{0}^{\pi} r^2 \cos \theta d\theta dz\) 
(c) \(\int_{0}^{3} \int_{0}^{2\pi} r \cos \theta d\theta dz\) 
(d) \(\int_{0}^{3} \int_{0}^{2\pi} r^2 \cos \theta d\theta dz\)
Voting Question 8: Which of the following is an iterated integral that will evaluate $\iiint_E (x-y) \, dV$ where $E$ is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$, above the $xy$-plane, and below the plane $z = y + 4$.

a) $\int_0^{2\pi} \int_0^2 \int_1^{y+4} (x-y) r^2 \sin \theta \, dz \, dr \, d\theta$

b) $\int_0^{2\pi} \int_0^3 \int_y^{y+4} (x-y) z \, dz \, dr \, d\theta$

c) $\int_0^{2\pi} \int_0^1 \int_0^{r \cos \theta} (r \cos \theta - r \sin \theta) r \, dr \, dz \, d\theta$

d) $\int_0^{2\pi} \int_0^3 \int_0^{4+r \sin \theta} (r \cos \theta - r \sin \theta) z \, dz \, dr \, d\theta$

e) none of the above

Example 3: Set up an integral in cylindrical coordinates that calculates the volume of the region bounded by the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$.
22.3.3 Spherical Coordinates

\[ \begin{align*}
\rho &\geq 0, \\
\leq \theta \leq \pi, \\
\text{and } -\pi \leq \phi \leq \pi.
\end{align*} \]

Converting Between Cartesian and Spherical Coordinates:

**Cartesian to Spherical:** \((x, y, z) \to (\rho, \theta, \phi)\):

\[
\begin{align*}
\rho &= , \\
\theta &= , \\
\phi &= .
\end{align*}
\]

**Spherical to Cartesian:** \((\rho, \theta, \phi) \to (x, y, z)\):

\[
\begin{align*}
x &= , \\
y &= , \\
z &= .
\end{align*}
\]

**Important Note:** Many physicists, engineers, scientists, and mathematicians reverse the roles of \(\theta\) and \(\phi\). Some even use \(r\) instead of \(\rho\) (because \(\rho\) represents charge or mass density). That is \(\phi\) denotes the polar angle in the xy-plane, and \(\theta\) denotes the angle from the positive z-axis. Our Calculus book uses the convention for \(\theta\) to denote the same thing as it does in polar coordinates and it lists the order of spherical coordinates as \((\rho, \text{polar angle}, \text{angle from positive z-axis})\). Your physics class may switch the order of these coordinates. I am very sorry for the confusion and hope that one day we will all have the same convention (maybe change polar coordinates to have \(\phi\) instead of \(\theta\)?). In the meantime, I believe in your maturity to overcome this!

**Voting Question 1:** What are the Cartesian coordinates of the point with spherical coordinates \((\rho, \theta, \phi) = (4, \pi, 0)\)?

(a) \((x, y, z) = (0, 0, -4)\)
(b) \((x, y, z) = (0, 0, 4)\)
(c) \((x, y, z) = (4, 0, 0)\)
(d) \((x, y, z) = (-4, 0, 0)\)
(e) \((x, y, z) = (0, 4, 0)\)

**Voting Question 2:** What are the spherical coordinates of the point with Cartesian coordinates \((x, y, z) = (0, 3, 0)\)?

(a) \((\rho, \theta, \phi) = (3, \frac{\pi}{2}, \pi)\)
(b) \((\rho, \theta, \phi) = (3, -\frac{\pi}{2}, \pi)\)
(c) \((\rho, \theta, \phi) = (3, \frac{\pi}{2}, \frac{\pi}{2})\)
(d) \((\rho, \theta, \phi) = (3, -\frac{\pi}{2}, \frac{\pi}{2})\)
(e) \((\rho, \theta, \phi) = (3, \pi, \frac{\pi}{2})\)
Example 1: Sketch the graph of $\rho = \sin(\phi) \cos \theta$

Voting Question 3: The graph of $\rho = 5$ is a
- (a) circle
- (b) sphere
- (c) cone
- (d) plane
- (e) cylinder

Voting Question 4: The graph of $\phi = \frac{\pi}{4}$ is a
- (a) circle
- (b) sphere
- (c) cone
- (d) plane
- (e) cylinder

Voting Question 5: Is the graph of $\theta = \frac{\pi}{4}$ the same graph in spherical coordinates as it is in cylindrical?
- (a) yes
- (b) no
- (c) I want to go home
22.3.4 Integration with Spherical Coordinates

Cartesian→Spherical:  \((x, y, z) \rightarrow (\rho, \theta, \phi)\):
- \(\rho^2 = x^2 + y^2 + z^2\)
- \(\theta = \arctan\left(\frac{y}{x}\right)\)
- \(\phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)\)

Spherical→Cartesian:  \((\rho, \theta, \phi) \rightarrow (x, y, z)\):
- \(x = \rho \sin \phi \cos \theta\)
- \(y = \rho \sin \phi \sin \theta\)
- \(z = \rho \cos \phi\)

Now let \(f(\rho, \phi, \theta)\) be a continuous function defined on a region \(R\) where \(R = \{(\rho, \phi, \theta) : a \leq \theta \leq b, g(\theta) \leq \phi \leq h(\theta), G(\phi, \theta) \leq \rho \leq H(\phi, \theta)\}\) where \(g, h, G, & H\) are continuous. Then the triple integral of \(f\) over \(R\) in spherical coordinates is

\[
\int\int\int_R f(\rho, \phi, \theta) dV =
\]

Voting Question 6: Which of the following integrals gives the volume of the unit sphere?

a) \(\int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin \phi d\rho d\phi d\theta\)

b) \(\int_0^{2\pi} \int_0^\pi \int_0^1 \rho d\rho d\phi d\theta\)

c) \(\int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin \phi d\rho d\phi d\theta\)

d) \(\int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin \phi d\rho d\theta d\phi\)

e) \(\int_0^{2\pi} \int_0^\pi \int_0^1 \rho d\rho d\theta d\phi\)

Example 2: Evaluate \(\int\int\int_R e^{-\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}} dV\) where \(R\) is the unit sphere.
Example 3: Set up an integral to find the volume of the snow cone made from the region inside the cone $\phi = \frac{\pi}{3}$ and inside the sphere $\rho = 5$.

Example 4: Set up an integral to find the volume of the snow cone made from the region inside the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $\rho = 5$.

Example 5: Set up an integral to find the volume of the region bounded by the plane $z = 25$ and the paraboloid $z = x^2 + y^2$. 
22.3.5 ICE 19 – Integration Using Cylindrical and Spherical Coordinates

1. Set up an integral to find the volume of the solid under the surface \( z = xy \), above the xy-plane, and within the cylinder \( x^2 + y^2 = 2x \).

2. Set up an integral that represents the volume above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere \( x^2 + y^2 + z^2 = z \).

\(^4\text{From Stewart’s Calculus}\)
23 Changing Variables in Integration

Ever wonder why when we change coordinate systems and then integrate we need to change:

Polar: Cylindrical: Spherical:

These are derived through a change of variable formula. In fact, we are very comfortable with changing variables, we discussed this when we did rotations and translations of axes AND we do this every time we integrate using

\[ \int_{a}^{b} f(g(x))g'(x)dx = \]

**Note:** If \( g \) is a one-to-one (that is, if \( g \) has an inverse function), then starting with \( \int_{a}^{b} f(x)dx \), we can transform \( \int_{a}^{b} f(x)dx = \)

In this section, we will be writing \( x, y, \) and \( z \) in terms of other new variables and the goal of these “transformations” is to make our lives easier when we integrate!

**Example 1:** If we write \( x \) and \( y \) as functions of new variables \( u \) and \( v \) so if \( x = 2u + 3v \) and \( y = u - v \), the following points in the \( u - v \) plane can be transformed into the \( xy \) plane by:

\( (0,0) \rightarrow (3,0) \rightarrow (3,1) \rightarrow (0,1) \rightarrow \)

**Example 2:** Find the transformation from the \( xy \) plane into the \( uv \) plane for \( u = 2x - 3y \) and \( v = y - x \). In other words, find equations for \( x \) and \( y \) in terms of \( u \) and \( v \).
23.1 The Jacobean

Recall the determinant of $\begin{vmatrix} a & b \\ c & d \end{vmatrix} =$ $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} =$

**Definition:** Given a transformation $T : x = g(u, v), y = h(u, v)$ where $g$ and $h$ are differentiable on a region in the $uv$ space, the *Jacobian determinant* or the *Jacobian* of the transformation is $J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} =$

If we have 3 variables, $J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} =$

**Example 3:** Compute the Jacobian for the transformation in Example 2.
**Theorem:** If $T$ is a one to one transformation which maps a bounded region $S$ in the $uv$ onto the bounded region $R$ in the $xy$, if $G$ is of the form $G(u, v) = (x(u, v), y(u, v))$, then $\iint_R f(x, y) dx dy =$

And we have the Triple Integral version: If $G(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$, then $\iiint_R f(x, y, z) dx dy dz =$

**Example 4:** Verify that $dV = r dz dr d\theta$. [We can also show $dV = \rho^2 \sin \phi d\rho d\phi d\theta$, but this is messier.]
Example 5: Let $R$ be the region in the first quadrant bounded by the parabolas $x = y^2$, $x = y^2 - 4$, $x = 9 - y^2$, and $x = 16 - y^2$. Evaluate $\int\int_R y^2 \, dA$.

\[\text{\textsuperscript{5}}\text{From Pearson’s \textit{Calculus: Early Transcendentals}}\]
23.2 ICE 20— Change of Variables

1. Evaluate the double integral $\iint_R \sin(x - y) \cos(x + y) dA$ where $R$ is the triangle with vertices $(0, 0), (\pi, -\pi)$, and $(\pi, \pi)$. 

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24 Vector Fields

24.1 Introduction to Vector Fields

What is a vector field?
A vector field is function whose domain is the set $\mathbb{R}^2$ or $\mathbb{R}^3$ and whose range is a set of ________. More formally,

**Definition:** Given $f$ and $g$ on a region $R \subseteq \mathbb{R}^2$, a *vector field* in $\mathbb{R}^2$ is a function $F$ that assigns each point in $R$ a vector __________________________. So we can write:

**Note:** A vector field is continuous or differentiable on a region $R$ if $f$ and $g$ are continuous or differentiable on $R$ respectively.

**Key Concept:** A vector field cannot be represented by a single curve or surface, instead we plot a sample of vectors to give us the general idea of the appearance of the vector field.

For example, consider the vector field defined by $F = \langle x, y \rangle$

\[ F = \langle x, y \rangle \]

**Idea:** for each point $P = (x, y)$ plot a vector with its tail at $P$ equal to the value of $F(x, y)$. So the length of our vector is ________.

**Example 1:** Sketch the vector field defined by $F = \langle 1 - y, 0 \rangle$ for $|y| \leq 1$.
**Voting Question 1:** Match the vector fields with the appropriate graphs. Let \( \mathbf{r} = <x, y> \).

1. \( \mathbf{F}_1 = \frac{\mathbf{r}}{||\mathbf{r}||} \)
2. \( \mathbf{F}_2 = \mathbf{r} \)
3. \( \mathbf{F}_3 = yi - xj \)
4. \( \mathbf{F}_4 = xj \)

(a) 1 and III, 2 and I, 3 and IV, 4 and II  
(b) 1 and IV, 2 and I, 3 and II, 4 and III  
(c) 1 and II, 2 and I, 3 and IV, 4 and III  
(d) 1 and II, 2 and IV, 3 and I, 4 and III  
(e) 1 and I, 2 and II, 3 and IV, 4 and III

**Note:** Vector Fields come up in electric fields, magnetic fields, force fields, and gravitational fields. In our class, we will only consider fields which are independent of time and we call these fields, **vector fields**.

We also have “scalar fields” which is just a fancy name for a function of space. These fields associate a number with a position in space. Examples of scalar fields include temperatures of plates and electrostatic potential.

We have already discussed one type of vector field, called **vector fields**. The \( \nabla f(x, y, z) \) points in the direction of greatest **gradient** of \( f(x, y, z) \).

**Voting Question 2:** The figure shows the vector field \( \mathbf{F} = \nabla f \). Which of the following are possible choices for \( f(x, y) \)? [Hint: Find \( \nabla f \).]

a) \( x^2 \)  
b) \( -x^2 \)  
c) \( -2x \)  
d) \( -y^2 \)

**Definition:** A vector field, \( \mathbf{F} \), that is the gradient of a scalar function is called a **conservative** vector field. In other words, there exists a scalar function \( \phi \) such that \( \nabla \phi \). We call \( \phi \) a **conservative scalar function** for \( \mathbf{F} \).

We will talk more about how we can test vector fields to find out whether or not they are conservative in a later section. Woo!
24.2 Curl and Divergence

**Definition:** Let \( \mathbf{F} = \langle f, g, h \rangle \) where \( f, g, \) & \( h \) are functions of \( x, y, \) & \( z \) for which the first partial derivatives exist. Then we define the *divergence* of \( \mathbf{F} \) of \( \sum F = \)  

and the *curl* of \( \mathbf{F} \) or \( \sum F = \)  

We call the divergence of \( \mathbf{F} \) a *derivative*. Using a slight but helpful abuse of notation, we see that \( \text{div} \mathbf{F} = \) .  

If we think of \( \mathbf{F} \) as the vector field of a flowing liquid, then \( \text{div} \mathbf{F} \) represents the net rate of change of the mass of the liquid flowing from the point \((x, y, z)\). We will learn more about divergence later.

We call the curl of \( \mathbf{F} \) a *derivative*. Notice \( \text{curl} \mathbf{F} = \) . \( \text{curl} \mathbf{F} = 0 \) if and only if \( \mathbf{F} \) is conservative –aka if \( \mathbf{F} \) is “nice.” Later we will talk about how it measure the net counter clockwise circulation around the axis that points in the direction of the curl.

**Example 2:** Find the divergence and curl of \( \mathbf{F} \) for \( \mathbf{F}(x, y, z) = \langle \cos x, \sin y, 3 \rangle \)
24.3 ICE 21 – Vector Fields

Sketch each of the following vector fields.

1. $-x\mathbf{i} - y\mathbf{j}$

2. $y\mathbf{i} - x\mathbf{j}$

3. $\frac{-x\mathbf{i} - y\mathbf{j}}{\sqrt{x^2 + y^2}}$

4. $y^2\mathbf{i} + x^2\mathbf{j}$
25 Line Integrals

When we evaluate $\int_{a}^{b} f(x)dx$, we are integrating over ________. We can generalize our integral by integrating over a ________ or ________ instead of our ________. This gives us something called a ________ integral. Although your text is correct when it says that they would be more accurately named ________ integrals.

**Definition:** If $f(x,y)$ is defined on a smooth curve $C$ given by $x = x(t), y = y(t), a \leq t \leq b$, then the line integral (with respect to arc length) of $f$ along a curve $c$ is

$$\int_{C} f(x,y)ds =$$

The 3-space version: $\int_{C} f(x,y,z)ds =$

**Definition:** The line integral of $f$ along $C$ with respect to $x$ is given by $\int_{C} f(x,y)dx =$

The line integral of $f$ along $C$ with respect to $y$ is given by $\int_{C} f(x,y)dy =$

Often the line integrals with respect to $x$ and $y$ occur together we usually write

$\int_{C} P(x,y)dx + \int_{C} Q(x,y)dy =$

**Note:** The choice of parametrization will not change the path of our curve which means our line integral will not change. And if we are taking a line integral with respect to ________ ________, switching the direction of the curve won’t change the value of the line integral.

**Example 1:** Evaluate $\int_{C} xy ds$, where $C : r(t) = 5ti + 4tj, 0 \leq t \leq 1$. 
Example 2: a) Evaluate $\int_C (x + y)ds$ where $C$ is the right half of the unit circle. Assume we move along the curve in the counter-clockwise direction.

b) Evaluate $\int_C (x + y)ds$ using a different parametrization.

c) Evaluate $\int_C (x + y)ds$ where $C$ and assume we move along the curve in the clockwise direction.
Example 3: Evaluate $\int_C xy^2 dx + 2x^2y dy$, where $C$ is the line segment from $(0,1)$ to $(4,3)$.

Note: Sometimes we need to break up our interval if our curve formula changes.

Example 4: Evaluate $\int_C (x + y) ds$ where $C = C_1 \cup C_2$ is the curve shown in the figure.
25.1 Line Integrals of Vector Fields

The motivation to evaluate line integrals of vector fields is __________! Recall, we used a dot product to compute the work done by a constant force \( \mathbf{F} \) from moving an object from point \( P \) to \( Q \) is \( W = \) __________ where \( \mathbf{D} \) is the displacement vector __________.

Now suppose we have a continuous force field (vector field) \( \mathbf{F} = < f, g, h > \) and we wish to compute the work done by \( \mathbf{F} \) in moving a particle along a smooth curve \( C : \mathbf{r}(t) = < P(t), Q(t), R(t) > \) for \( a \leq t \leq b \). When we partition our curve into subintervals and make our subintervals, \( \Delta s_i \) super small, then as our particle moves along our curve it moves approximately in the direction of the unit tangent vector of our curve at the point along the partition. Thus the work is approximately \( \mathbf{F} \cdot T \Delta s \).

Thus the work done is \( \int_C \mathbf{F} \cdot T ds = \) which is also equal to

**Definition:** Let \( \mathbf{F} = < f, g, h > \) be a continuous vector field, then the line integral of \( \mathbf{F} \) along \( C \) (where \( C \) is given by the vector function \( \mathbf{r}(t) \) for \( a \leq t \leq b \) is

**Example 5:** A particle travels along the helix \( C \) given by \( \mathbf{r}(t) = \cos(t)i + \sin(t)j + 2tk \) and is subject to the force \( \mathbf{F}(x, y, z) = xi + zj - xyk \). Find the total work done on the particle by the force for \( 0 \leq t \leq 3\pi \).
25.2 ICE 22– Line Integrals

1. Evaluate \( \int_C xy\,ds \) where \( C \) is the circle with radius 2 on the oriented path from \((2,0)\) to \((0,2)\).

2. Evaluate \( \int_C xy\,ds \) where \( C \) is the circle with radius 2 on the oriented path from \((0,2)\) to \((2,0)\).

3. What does this tell you about the relationship between \( \int_C f\,ds \) and \( \int_{-C} f\,ds \)?

4. Evaluate \( \int_C xy\,ds \) where \( C \) is the line from \((2,0)\) to \((0,2)\).
26 The Fundamental Theorem of Line Integrals

Remember the Grand ’ol Fundamental Theorem of Calculus?

Fundamental Theorem of Line Integrals: If C is a piecewise smooth curve given by \( \vec{r}(t) \) on \( a \leq t \leq b \) and \( f \) is a differentiable function on \( C \) with gradient \( \nabla f \), then

\[
\int_C \nabla f \cdot d\vec{r} = \]

Voting Question 1: The line integral of \( F = \nabla f \) along one of the paths shown below is different from the integral along the other two. Which is the odd one out?

a) C1
b) C2
c) C3

Key Ideas: If we are given \( \int_C f \, d\vec{r} \), we can’t necessarily use our theorem (we need __________... but this theorem gives us the motivation to study the next few things...

26.1 Curve Talk with Dr. Harsy: Curve Adjectives:

Positively Oriented Curve: Simple Curve: Closed Curve:

Definition: A set \( D \) is connected if any two points in \( D \) can be connected by a piecewise curve that stays within \( D \). \( D \) is __________ __________ if every simple path in \( D \) encloses points only in \( D \).
Definition: Given a connected set \( D \) we say \( \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \) is independent of path in \( D \) if for any two points, say \( A \) & \( B \) in \( D \), the line integral will have the same value for any path from \( A \) to \( B \) that stays in \( D \) AND is positively oriented.

AKA: The parameterization __________ __________ __________!!
So “path independent vector fields” have the property that for any two points, \( A \) and \( B \), the line integral from \( A \) to \( B \) is independent of path.
**Example 1:** Suppose $C$ is a curve that starts and ends at the same point (called a closed curve). So $\mathbf{r}(a) = \mathbf{r}(b)$ Then $\int_C \nabla f \cdot d\mathbf{r} =$

**Theorem:** If vector field $\mathbf{F}$ is continuous on an open connected set $D$, then $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is independent of path in $D$ if and only if $\mathbf{F} = \nabla f$ for some scalar function $f$.

Recall we call such a scalar function $f$ a function for $\mathbf{F}$. And a vector field, $\mathbf{F}$, that is the gradient of a scalar function is called a vector field.

Thus..

**Theorem:** The following are equivalent:

- $\mathbf{F}$ is a vector field.
- $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is
- $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = 0$ for any path $C$.

**Voting Question 2:** A gradient field is path independent.

a) True, and I am very confident!

b) True, but I am not very confident...

c) False, and I am very confident!

d) False, but I am not very confident...

**Voting Question 3:** Path independent vector fields are gradient fields.

a) True, and I am very confident!

b) True, but I am not very confident...

c) False, and I am very confident!

d) False, but I am not very confident...
26.2 Testing For Conservative Vector Fields

Testing for Conservative Vector Fields Theorem: Let \( \mathbf{F} = \langle f, g, h \rangle \) be a continuous vector field defined on a connected set \( D \), then \( \mathbf{F} \) is a conservative vector field (aka: we can find a \( \frac{\partial g}{\partial y} = \frac{\partial h}{\partial z} = \frac{\partial f}{\partial x} \) function \( \alpha \) s.t. \( \mathbf{F} = \nabla \alpha \)) if and only if

In other words, \( \int \mathbf{F} = \nabla \alpha \)

Note: If we have a vector field in \( \mathbb{R}^2 \), we only need:

Example 1: Determine which vector fields are conservative.

a) \( \mathbf{F}(x, y) = e^{2y} \mathbf{i} + (1 + 2xe^{2y}) \mathbf{j} \)

b) \( \mathbf{F}(x, y, z) = \langle 2xy - z^2, x^2 + 2z, 2y - 2xz \rangle \)

c) \( \mathbf{F}(x, y, z) = xy^2z^3 \mathbf{i} + 2x^2yz^3 \mathbf{j} + 3x^2y^2z^2 \mathbf{k} \)
26.3 Finding Potential Functions

Example 2: Find a potential function for $\mathbf{F} = e^{2y} \mathbf{i} + (1 + 2xe^{2y}) \mathbf{j}$.

Example 3: Evaluate $\int_C (e^{2y} \mathbf{i} + (1 + 2xe^{2y}) \mathbf{j}) \cdot d\mathbf{r}$ where $C$ is the curve below.
Example 4: Consider \( \mathbf{F} = < 2xy - z^2, x^2 + 2z, 2y - 2xz >. \)

a) Find a potential function for \( \mathbf{F} \). What do we need to check first?

b) Compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \) for the curve shown below.
27 Green’s Theorem

The rest of Calc III will be introducing different variations of our wonderful fundamental theorem of calculus. All of these generalizations come up in physic topics like electricity, magnetism, and fluid flow. So far we have discussed the fundamental theorem of line integrals. What did we need to have in order to use the fundamental theorem of line integrals?

We now discuss a variation of the Fundamental Theorem of Line Integrals in which we don’t need our vector field to be \text{_______________________}.

**Green’s Theorem:** Let \( C \) be a positively oriented piecewise smooth, simple, closed curve in the plane and let \( D \) be the region bounded by \( C \). If \( P \) and \( Q \) have continuous partial derivatives on an open region containing \( D \), then

\[
\oint_C (P\,dx + Q\,dy) = \iint_D \left( \frac{	riangledown Q}{	riangledown P} \right)\, dA
\]

**Key:** Green’s Theorem can replace a difficult line integral by an easier double integral or vise versa.

**Notation:** When our curve is closed, we can write \( \oint_C \) as \___________.

If \( \mathbf{F} \) is conservative, Green’s Theorem says that the line integral along a closed curve is 0. Green’s Theorem is like the FTC because it relates an integral of a “derivative” on a region to its “antiderivative” on a boundary.

**Example 1:** Evaluate the line integral \( \oint_C y\,dx - x\,dy \) where \( C \) is the unit circle centered at the origin directly without using Green’s Theorem.

Directly:
Example 2: Evaluate the line integral \( \int_C y \, dx - x \, dy \) where C is the unit circle centered at the origin using Green’s Theorem.

Green’s:

Example 3: Evaluate the line integral \( \int_C (x^4 + 2y) \, dx + (5x + \sin y) \, dy \) where C is boundary of the “unit diamond.”

\[
\begin{array}{c}
\text{y} \\
\hline \\
\text{x}
\end{array}
\]

Notice, Green Theorem applies to a positively oriented piecewise smooth, simple, closed curve, regardless if our vector field is \textbf{closed} curve!!

\textbf{Note:} If \( Q_x - P_y = 0 \), \( \mathbf{F} \) is conservative. Otherwise \( Q_x - P_y \) measures how “not conservative” \( \mathbf{F} \) is. Let’s consider \( \mathbf{F}(x, y, z) = Q \mathbf{j} \)

If \( Q_x < 0 \):

\[
\begin{array}{c}
\text{y} \\
\hline \\
\text{x}
\end{array}
\]

If \( Q_x > 0 \):

\[
\begin{array}{c}
\text{y} \\
\hline \\
\text{x}
\end{array}
\]
Now consider $\mathbf{F}(x, y, z) = Pi$

If $P_y < 0$:

If $P_y > 0$:

If $Q_x - P_y > 0$, our rotation is _____________

If $Q_x - P_y < 0$, our rotation is _____________

If $Q_x - P_y = 0$, our rotation is _____________

So $Q_x - P_y$ measures the tendency to rotate counterclockwise!

Since $Q_x - P_y$ is determined by the _______ of a vector field, _______$\mathbf{F}$ measures the net counterclockwise circulation of a vector field.

**Example 4:** For each field use the sketch to decide whether the curl at the origin points up down or is the zero vector. Then check your answer using the coordinate definition of curl.

a) $\mathbf{F}(x, y) = xi + yj$

b) $\mathbf{F}(x, y) = yi - xj$

c) $\mathbf{F}(x, y) = -(y + 1)i$
27.1  ICE 23 – Green’s Theorem

1. Evaluate the integral $\int \int _{R} 2y \, dA$ where $R$ is the region bounded by a triangle with vertices $(0, 0), (1, 0), \text{and} (0, 3)$ without using any theorems.

2. Evaluate $\oint \sqrt{1 + x^3} \, dx + 2xy \, dy$ where $C$ is the positively oriented triangle with vertices $(0, 0), (1, 0), \text{and} (0, 3)$. Use Green’s Theorem and your answer from part 1.
28 Curl and Divergence Revisted

28.1 More About Vector Fields

Let’s go back to the fundamental theorem of line integrals. In order to use this, we want a ________ vector field. In other words, we wonder whether $\mathbf{F}$ is a gradient field or whether or not it is path independent or whether or not it has rotation.

Recall, when we have a gradient field, the vectors are always ________ to the contours.

Voting Question 1: Which of the following vector fields cannot be a gradient vector field?

Example 1: The figure below shows a path of a vector field $\mathbf{F}$ along with 3 curves $C_1, C_2, C_3$. Remember $\int_C \mathbf{F} \cdot d\mathbf{r}$ is calculating the work done in moving a particle through a force field along curve $C$.

a) What is the sign of $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$?

b) What is the sign of $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$?

c) What is the sign of $\int_{C_3} \mathbf{F} \cdot d\mathbf{r}$?
Voting Question 2: Which of the following explains why this vector field is not a gradient vector field?

(a) The line integral from \((-1, 1)\) to \((1, 1)\) is negative.
(b) The circulation around a circle centered at the origin is zero.
(c) The circulation around a circle centered at the origin is not zero.
(d) None of the above

Voting Question 3: The Figure below shows the vector field \(\nabla f\), where \(f\) is continuously differentiable in the whole plane. The two ends of an oriented curve \(C\) from \(P\) to \(Q\) are shown, but the middle portion of the path is outside the viewing window. The line integral \(\int_C \nabla f \cdot dr\) is

(a) Positive
(b) Negative
(c) Zero
(d) Can’t tell without further information

Voting Question 4: Which of the diagrams contain all three of the following: a contour diagram of a function \(f\), the vector field \(\nabla f\) of the same function, and an oriented path \(C\) from \(P\) to \(Q\) with \(\int_C \nabla F \cdot dr = 60\)?

(i) 
(ii) 
(iii) 
(iv)
29 Parametric Surfaces

Just like in the 2-dimensional case, in higher dimensions, it can be beneficial to write equations using parameters.

Given a surface \( z = G(x, y) \), we can always parameterize it by

\[
\begin{align*}
x &= \\
y &= \\
z &= 
\end{align*}
\]

We can also generalize this. If the surface is written as \( x = G(y, z) \), we can parameterize it by

\[
\begin{align*}
x &= \\
y &= \\
z &= 
\end{align*}
\]

Unfortunately this can only work if we can write \( x, y, \) or \( z \) as an explicit function of the other variables and sometimes we can’t do this which means we turn to different parameterizations.

**Example 1:** Parameterize the surface represented by the top half of the cone \( z^2 = 3x^2 + 3y^2 \).

a) Note that since we only want the top half, we can write \( z \) as an explicit function of \( x \) and \( y \).

b) Find another representation for this surface.

**Example 2:** Parameterize the sphere \( x^2 + y^2 + z^2 = 4 \).
29.1 Recognizing Surfaces From Parametric Equations

Match the following equations numbered 1-6 with the graphs on the next page. In each case determine what the grid curves are when \( u \) or \( v \) are constant. Eliminate the parameters where possible to give a rectangular equation for each surface.

1. \( \mathbf{r}(u,v) = < \sin u, \cos u, v > \)
   
   If \( u \) is constant, the grid curves are:
   
   If \( v \) is constant, the grid curves are:
   
   The surface number is:

2. \( \mathbf{r}(u,v) = < \sin u, v, \cos u > \)
   
   If \( u \) is constant, the grid curves are:
   
   If \( v \) is constant, the grid curves are:
   
   The surface number is:

3. \( \mathbf{r}(u,v) = < v \sin u, v \cos u, v > \)
   
   If \( u \) is constant, the grid curves are:
   
   If \( v \) is constant, the grid curves are:
   
   The surface number is:

4. \( \mathbf{r}(u,v) = < v, v \cos u, v \sin u > \)
   
   If \( u \) is constant, the grid curves are:
   
   If \( v \) is constant, the grid curves are:
   
   The surface number is:

5. \( \mathbf{r}(u,v) = < v \sin u, v \cos u, v^2 > \)
   
   If \( u \) is constant, the grid curves are:
   
   If \( v \) is constant, the grid curves are:
   
   The surface number is:

6. \( \mathbf{r}(u,v) = < v \sin u, v \cos u, u > \)
   
   If \( u \) is constant, the grid curves are:
   
   If \( v \) is constant, the grid curves are:
   
   The surface number is:

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30 Surface Integrals

Recall from 21.4 Definition: Given a smooth parametric surface $S$ given by the equation $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ (where $S$ is covered just once as the parameters range through the domain of $S$), then the surface area of $S$ is given by

$$A(S) = \int_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

Note $\mathbf{r}_u = \langle x_u, y_u, z_u \rangle$ and $\mathbf{r}_v = \langle x_v, y_v, z_v \rangle$.

**Explicit Case:** To find the Area of a surface represented explicitly by $z = g(x, y)$, we evaluate

$$A(S) = \int_R \sqrt{1 + (g_x)^2 + (g_y)^2} \, dA$$

A line integral generalizes the ordinary definite integral by integrating over a ______. Now we discuss surface integrals which generalize ______ integrals by integrating over a ______.

**Parametric Case:** Now if a surface $S$ is given by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, the **surface integral of a function $f$ over $S$** is given by

$$\iint_G f(x, y, z) \, dS = \iint_R f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

**Explicit Case:** Given a surface explicitly given by $z = g(x, y)$ with domain $R$. Let $f(x, y, z)$ be a function defined on $S$ with continuous first-ordered partial derivatives, then the **surface integral of $f$ over $G$** is

$$\iint_G f(x, y, z) \, dS = \iint_R f(x, y, g(x, y)) \sqrt{1 + (g_x)^2 + (g_y)^2} \, dA$$
Often we think of \( f(x, y, z) = \) ________________ and can generalize it:
Example 1: Evaluate the surface integral $\int\int_G zdS$ where $G$ is the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 4$. (Note $z = x^2 + y^2$ is a 1 sided surface.)

Example 2: Set up the surface integral $\int\int_G x^2dS$ where $G$ is the sphere with radius 1.
**Key Technique:** If $G$ is a piece-wise surface, we must “break up” our surface into parts and integrate over each part of the surface.

**Example 3:** Evaluate the surface integral $\iint_G zdS$ where $G$ is the part of the tetrahedron bounded by the coordinate planes and the plane $4x + 8y + 2z = 16$. 

![Diagram showing the tetrahedron and its footprints on the coordinate planes.](224)
30.1 Surface Integrals of Vector Fields/ Flux

**Definition:** If \( \mathbf{F}(x, y) \) denotes the velocity of the fluid at \((x, y)\) in a region \( S \) (in xy plane) as it crosses its boundary curve \( C \), the *flux* of \( \mathbf{F} \) is the net amount of fluid leaving \( S \) per unit of time.

Consider a region \( R \) enclosed by curve \( C \).
So the *flux* (total amount of fluid leaving \( R \)) of \( \mathbf{F} \) across the curve \( C \) is \( \oint_C \).

Now suppose we want calculate the flux or flow of a vector field \( \mathbf{F} \) across a surface.
First we need to specify an orientation of our surface with a normal vector. That is, we need a 2-sided surface. (Note in Ex 1, \( z = x^2 + y^2 \) is a 1 sided surface.)

**Surface Talk with Dr. Harsy:**
For a closed surface, the “positive orientation” is ____________
If the normal vector for a surface varies continuously, we say our surface is ____________.

**Definition:** Given a continuous vector field \( \mathbf{F} = \langle P, Q, R \rangle \) defined on an oriented closed smooth surface \( G = z = g(x, y) \) with domain \( R \), the surface integral of \( \mathbf{F} \) over \( S \) or Flux Integral is given by

\[
\int_S \mathbf{F} \cdot dS = \int_S \mathbf{F} \cdot \mathbf{n}dS
\]

Our normal vector \( \mathbf{n} \) can be written as ______________ which means we also have that

\[
\int_G \mathbf{F} \cdot \mathbf{n}dS =
\]

**Parametric Case:** For a surface given by \( \mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle \), the surface integral of \( \mathbf{F} \) over \( S \) is given by

\[
\int_S \mathbf{F} \cdot dS = \int_S \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)dS
\]

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Example 4: Calculate the flux of $\mathbf{F}(x, y, z) = i + 2j + 3k$ for the following oriented surfaces.

a) A unit square in the xy-plane oriented upwards.

b) A unit square in the yz-plane oriented towards the positive x-axis.

c) A unit square in the xz-plane oriented towards the negative y-axis.

Example 5: Calculate the flux of $\mathbf{F}(x, y, z) = (y)j$ across the part of the plane $4x + 8y + 2z = 16$ in the first octant. Note: Use Example 2.
31 Stokes’ Theorem

**Example 1:** Consider the vector field given by \( \mathbf{F}(x, y, z) = yi - xj + zk \) and the surface \( S \) that is the hemisphere \( x^2 + y^2 + z^2 = 1, \ z \geq 0 \) oriented upward.

1) Evaluate \( \text{curl} \mathbf{F} \)

2) Sketch the surface \( S \) and a normal vector orienting the surface.

3) What is the boundary of \( S \)? Let’s call it \( \partial S \) or \( C \).

4) Parameterize \( C \) with *positive orientation*.

5) Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \). [Why must we use our parameterization of \( C \)?]

6) Now evaluate \( \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} \). Recall, \( \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} dS \).
7) What do you notice about the answers you got in part 5 and 6?

This is an example of Stokes’ Theorem:

**Stokes’ Theorem** Let S be an oriented piecewise smooth surface that is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let \( \mathbf{F} \) be a vector field whose components have continuous partial derivatives around S. Then,

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl}\mathbf{F} \cdot \mathbf{n}dS
\]

**Example 2:** Let \( \mathbf{F} = \langle e^{-x}, e^x, e^z \rangle \) and let C be the boundary of the part of the plane \( 2x + y + 2z = 2 \) in the first octant oriented counterclockwise as viewed from above. Use Stokes’ Theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) using the following steps.

1) Sketch the curve C, including orientation.

2) Why would calculating the line integral over C not be ideal?

3) Instead of calculating \( \int_C \mathbf{F} \cdot d\mathbf{r} \), let’s calculate:

First we need to describe a surface S whose boundary is C.

Then we determine an equation \( z = g(x, y) \) and a domain D for that surface S:

Next we determine \( \text{curl}\mathbf{F} \)

And also the normal vector \( \mathbf{n} \):

The we evaluate \( \iint_S \text{curl}\mathbf{F} \cdot \mathbf{n}dS \). (Remember by Stokes’ Theorem, this then equals \( \int_C \mathbf{F} \cdot d\mathbf{r} \).)
31.1  ICE 24– Stokes’ Theorem

1. Use Stokes’ Theorem to evaluate the $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = <2y, -z, x>$ where $C$ is the boundary of the square $|x| \leq 1, |y| \leq 1$ in the xy plane $(z = 0)$. Do this by calculating $\iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} dS$. Hint: Use the square as your surface. If I have a flat square in the x-y plane, what is $\mathbf{n}$?

2. Use Stokes’ Theorem to evaluate the surface integral $\iint_S \text{curl} \mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F} = yz \mathbf{i} + xz \mathbf{j} + xy e^z \mathbf{k}$ and $S$ is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$. Note $\text{curl} \mathbf{F} = <xe^z - x, y - ye^z, 0>$ so do this by calculating $\oint_C \mathbf{F} \cdot d\mathbf{r}$. Hint: Use the trig identity: $\cos^2 t - \sin^2 t = \cos(2t)$. 

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32 Gauss’s Divergence Theorem

32.1 Vector Forms of Green’s Theorem

Curl and divergence operators allow us to rewrite Green’s theorem. Suppose we have a plane region $D$ with boundary curve $C$ and function $P$ and $Q$ which satisfy Green’s Theorem. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int P\,dx + Q\,dy.$$ 

If our curve is parameterized using arc length, say $x = x(s), y = y(s)$, then our tangent vector and normal vectors for our curve are given by:

$$\mathbf{T} = \mathbf{N} =$$

Now

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds =$$

This is sometimes called Gauss’s Divergence Theorem in the plane. Restated in terms of Flux, this means that the flux across $C$ can be calculated as $\oint_C \mathbf{F} \cdot \mathbf{n} ds$.

32.2 Gauss’s Divergence Theorem

We just discussed Gauss’s Theorem in the Plane, now suppose we consider a surface that is the boundary of a solid region $S$. We denote the boundary of our solid $S$ as $\partial S$. Gauss’s Theorem in the plane says that $\oint_{\partial S} \mathbf{F} \cdot \mathbf{n} ds = \iint_S \text{div} \mathbf{F} \, dA$

**Gauss’s Divergence Theorem:** Let $S$ be a simple solid with positive orientation and boundary $\partial S$. And let $\mathbf{F}$ be a vector field with continuous first partial derivatives around $S$. Then

$$\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \int \int_S \text{div} \mathbf{F} \, dA$$

**Note:** The above integral calculates the flux of $\mathbf{F}$ across $S$.

**Warning:** The divergence theorem can only work with simple solid regions. If we want to calculate the surface integral $\int \int_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = \langle xy, yz, zx \rangle$ and $S$ is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ and has upward orientation. Why can’t we use the divergence theorem?

In this situation, we need to set up an integral to calculate this surface integral.
**Example 1:** During our Surface Integral Notes, we calculated the flux of \( \mathbf{F}(x, y, z) = (y)\mathbf{j} \) across the part of the plane \( 4x + 8y + 2z = 16 \) in the first octant. Check our answer using Gauss’s Divergence Theorem and use the fact that the volume of a tetrahedron is \( \frac{1}{3}h(\text{area of the base}) \).

**Example 2:** Consider \( \mathbf{F} = \langle \frac{xy^2}{2}, \frac{y^3}{6}, zx^2 \rangle \) over the surface \( S \) where \( S \) is the cylinder \( x^2 + y^2 = 1 \) capped by the planes \( z = \pm 1 \). What is the flux of \( \mathbf{F} \) over \( S \)?
**Example 3:** Let $S$ be any solid sphere. Consider the two 3-dimensional vector fields given by $\mathbf{F}(x, y, z) = \langle 8x + 3y, 5x + 4z - 2y, 9y^2 - \sin x + 7z \rangle$ and $\mathbf{G}(x, y, z) = \langle 12y + 8z, e^z + \sin x + 9y, xy^2e^{xy} + 4z \rangle$. Without performing any integration, show that $\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iint_{\partial S} \mathbf{G} \cdot d\mathbf{S}$.

**Example 4:** Determine $\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\partial S$ is the outward oriented boundary of the surface shown in the figure below:

![Diagram](image)

**Note:** Example 4, if the divergence wasn’t constant, the limits of integration would be very tough to set up. What would we have to do?
32.3  ICE 25- Gauss’ Divergence Theorem

1. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ (that is calculate the flux of $\mathbf{F}$) across $S$ for $\mathbf{F} = \langle 2x^3 + y^3, y^3 + z^3, 3y^2z \rangle$ and where $S$ is the surface of the bounded by the paraboloid $z = 1 - x^2 - y^2$ and the $xy$ plane.

Now set up an integral to calculate this not using the divergence theorem.
2. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ (that is calculate the flux of $\mathbf{F}$) across $S$ for $\mathbf{F} = <x^3, y^3, z^3>$ and where $S$ is the surface of the solid bounded by the cylinder $1 = x^2 + y^2$ and the planes $z = 0$ and $z = 2$.

3. Note that the divergence theorem can only work with simple solid regions. If we want to calculate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = <xy, yz, zx>$ and $S$ is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ and has upward orientation. Why can't we use the divergence theorem? Set up an integral to calculate this surface integral.
33 Vector Calculus Flow Chart
Line Integrals:
\[ \int_C f \, ds, \quad \int_C f \, dx, \quad \int_C f \, dy, \quad \int_C f \, dz \]

Parameterize \( C \) using \( c(t) \):
\[ x = x(t), \quad dx = x'(t) \, dt \]
\[ y = y(t), \quad dy = y'(t) \, dt \]
\[ z = z(t), \quad dz = z'(t) \, dt \]

Parameterize if \( \mathbf{F} = \langle P, Q \rangle \) use Green's Thm:
\[ \iint_D Q_x - P_y \, dA \]

If \( \mathbf{F} = \langle P, Q \rangle \) is conservative:
Find potential \( f \):
\[ f(c(b)) - f(c(a)) \]

\[ \int_C \mathbf{F} \cdot \mathbf{dr} \]

Stokes' Theorem:
\[ \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} \]

Surface Integrals:
\[ \iint_S f \, dS \text{ or } \iint_S \mathbf{F} \cdot d\mathbf{S} \]

Divergence Thm:
\[ \iiint_V \text{div}(\mathbf{F}) \, dV \]

Fundamental Thm of Line Integrals:
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = f(c(b)) - f(c(a)) \text{ where } \mathbf{F} = \nabla f \]

Double Integrals:
\[ \int \int_R dA \]
34 Practice Problems and Review for Exams

The following pages are practice problems and main concepts you should master on each exam. Core concepts will have heavier weight than regular concepts.

These problems are meant to help you practice for the exam and are often harder than what you will see on the exam. You should also look over all ICE sheets, Homework problems, Quizzes, and Lecture Notes. Make sure you can do all of the problems in these sets.

Disclaimer: The following lists are topics that you should be familiar with, and these are problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

I will post the solutions to these practice exams on our blackboard site. If you find any mistakes with my solutions, please let me know right away and feel free to email at any hour.

Good Luck,
Dr. H
MA 250 – Practice Problems for Mastery Exam 1

Review all examples from Notes, Homework, Ice sheets, and Quizzes. Know basic theorems and definitions. Make sure you can do all of those problems. You can also practice with the problems below. Note you can use a calculator on all problems including sketching polar curves. Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

Here are the 5 Mastery Concepts:

1. Parametric Equations and Lines in space (Problems 1-4)
   - Know how to differentiate parametric equations
   - Know how to sketch parametric equations
   - Know how to write equations of lines in space

2. Planes and Vector Valued Functions (Problems 5-10)
   - Know how to find the unique equation of a plane that is perpendicular to a curve at a given point (use Tangent Vector)
   - Know how to find position, velocity, and acceleration functions and their relationships with each.
   - Know how to show a vector valued function lies on a given surface.

3. Curvature (Problem 11)
   - I will give you the definition of curvature, you will just have to compute it.

4. Surfaces (Problems 12,13,15)
   - Know how to recognize surfaces from an equation and vise versa.
   - Be able to give verbal descriptions of traces of a surface in the planes $x = k$, $y = k$, and $z = k$.
   - Be able to create and use a contour map to recognize a surface.

5. Graphical Analysis of Partial Derivatives (Problems 14, 16-19)
   - Know how to determine the signs of higher partial derivatives given a surface
   - Be able to determine the signs of higher partial derivatives given a contour map
   - Know how to approximate partial derivatives given a table of values
   - Know how to determine the signs of partial derivatives given an equation and be able to explain what this means for the rates of change of the problem.
1. Given the parametric curve \( x = t^2 + 1, \ y = e^t - 1 \).
   a) Find \( \frac{dy}{dx} \)
   b) Find \( \frac{d^2y}{dx^2} \)

2. You are at the point \((-1, -3, -3)\) standing upright and facing the \(xz\)-plane. You walk 2 units forward and turn left and walk for another 2 units. What is your final position?

3. Find parametric and symmetric equations of the line passing through the points \((-1, 0, 5)\) and \((4, -3, 3)\).

4. Find the equation of the tangent line for \( \mathbf{r}(t) = (12t^2, -4 \cos t) \) at \( t = 0 \).
5. Find the equation of the plane perpendicular to the curve \( r(t) = \langle \cos(t), \sin(t), t \rangle \) at \( t = \frac{\pi}{2} \).

6. Find a vector orthogonal to both \( \langle 1, -1, 0 \rangle \) and \( \langle 1, 2, 3 \rangle \) in order to find the equation of the plane that contains both vectors.

7. Find an equation of a plane that passes through the point \((1, 3, 4)\) and is parallel to the plane \( 6x - 2y + 9z = 3 \).

8. Given any two vectors \( \mathbf{u} \) and \( \mathbf{v} \), explain why \((\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0\).
9. Find the position and velocity vectors if \( \mathbf{a} = 12t^2, -4 \cos t \) and \( \mathbf{v}(0) = -\mathbf{i} \) and \( \mathbf{r}(0) = 2\mathbf{i} + \mathbf{j} \).

10. Find the position function of a particle given that the velocity function is \( \mathbf{v}(t) = 2t, \cos t \) with initial position \( \mathbf{r} = \mathbf{i} \).

11. Find the curvature of the curve given by \( x = t, y = \ln(t) \).
12. Circle the equation that generates the surfaces shown below.

\[ x^2 + y^2 + z^2 = 1 \quad x = z^2 - y^2 \quad z = 1 - x^2 \quad z = 1 - y^2 \]

\[ x^2 + z^2 - x^2 = 1 \quad x = z^2 + y^2 \quad x = 1 - y^2 \quad y = 1 - x^2 \]

\[ x^2 = y^2 + z^2 \quad x^2 - y^2 - z^2 = 1 \quad x = 1 - z^2 \quad y = 1 - z^2 \]

13. Know how to recognize the graphs of \( z = x^2 + y^2 \), \( z = x^2 - y^2 \), \( z = -x^2 - y^2 \), and \( z = \frac{1}{x^2 + y^2} \).

14. Let \( f(z, y) = \ln(x - y) \). Find the mixed partial derivatives of \( f_{xy} \) and \( f_{yx} \).

15. Sketch the contour map for \( f(x, y) = \sqrt{x + y} \).
16. Following the contour map for the function \( f(x, y) \) determine the signs of \( f_x(P) \), \( f_y(P) \), \( f_{xx}(P) \), \( f_{yy}(P) \), and \( f_{xy}(P) \).

17. Consider the surface below which represents \( f(x, y) \) Determine the signs of \( f_{xx}(P) \) and \( f_{yy}(P) \)

18. Suppose that the price \( P \) (in dollars) to purchase a used car is a function of its original cost \( C \) (in dollars) and its age \( A \) (in years). A possible model for this function is \( P = Ce^{-0.02A} \). What is the sign of \( \frac{\partial P}{\partial C} \)? Explain in one sentence what your previous answer above means in the context of the cost of the used car.

19. The following table gives values of the function \( f(x, y) \). (The row headings are \( x \)-values and the column headings are \( y \) values.) Use the table to approximate \( f_x(5, 4) \) and \( f_y(5, 4) \)

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<th>6</th>
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<td>17</td>
<td>18</td>
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<td>20</td>
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<td>57</td>
<td>59</td>
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</tr>
</tbody>
</table>

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MA 250 – Practice Problems for Mastery Exam 2

Review all examples from Notes, Homework, Ice sheets, and Quizzes. Know basic theorems and definitions. Make sure you can do all of those problems. You can also practice with the problems below. Note you can use a calculator on all problems including sketching polar curves. Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

Here are the 5 Mastery Concepts:

1. Polar Curves
   - Be able to convert between polar and Cartesian equations.
   - Be able to sketch a polar curve
   - Find the slope of the tangent line at a point on a polar curve.
   - Find the area bounded by a polar curve.

2. Partial Derivatives (Directional Derivatives, Chain Rule, Gradient)
   - Use the Chain Rule to compute partial derivatives.
   - Use the Chain Rule to calculate implicit derivatives.
   - Compute directional derivatives.
   - Find the direction of the maximum and minimum rate of change.

3. Limits
   - Prove a limit does not exist by exhibiting two paths of approach with different limiting behavior.
   - Prove a limit exists by appealing to continuity.
   - Prove a limit exists by separating the variables.
   - Use Polar Coordinates to prove a limit exists or does not exist.

4. Optimization and Classifying Extrema
   - Find critical points of a function.
   - Classify critical points of a function.
   - Find extreme values of a function on a closed, bounded domain.

5. Lagrange Multipliers
   - Use Lagrange Multipliers to find max/mins given a constraint.
1. Find the area inside one of the leaves in the curve $r = 4 \cos(2\theta)$

2. Find the area inside both $r = 4 - 2 \cos(\theta)$ and $r = 6 \cos \theta$

3. Find the Cartesian coordinates of the point whose polar coordinates are $(-2, \frac{3\pi}{2})$

4. Find the slope of the tangent line in terms of $\theta$ for $r = 2 + 2 \sin \theta$. For what $\theta$ is the tangent line horizontal?
5. The part of a tree normally sawed into lumber is the trunk, a solid shaped approximately like a right circular cylinder. If the radius of the trunk of a certain tree is growing \( \frac{1}{2} \) in/year and the height is increasing 8in/year, how fast is the volume increasing when the radius is 20 inches and the height is 400 inches?

6. Sand is pouring onto a conical pile in such a way that at a certain instant the height is 100 inches and increasing at 3 in/min and the base radius is 40 inches and increasing at 2 in/minute. How fast is the volume increasing at that instant?

7. Find \( \frac{\partial z}{\partial t} \) if \( z = x^2 y, x = 2s + t, \) and \( y = 1 - st^2. \)
Evaluate the following limit or explain why it do not exist. Justify your answer.

8. \[ \lim_{(x,y) \to (0,0)} \frac{|xy|}{xy} \]

9. \[ \lim_{(x,y) \to (0,0)} \frac{2xy}{x^2 + 2y^2} \]

10. \[ \lim_{(x,y) \to (0,0)} \frac{y(e^x - 1)}{x} \]

11. \[ \lim_{(x,y) \to (0,0)} \frac{yx}{\cos(x)} \]
12. Find the directional derivative of \( h(x, y, z) = xyz \) at the point \( P = (2, 1, 1) \) in the direction of \( \mathbf{v} = <2, 1, 2> \).

13. What is the direction of the maximum increase of \( f(x, y, z) = xe^y \) at the point \( P = (2, 0, 2) \)?

14. Let \( f(x, y, z) = x^3 + 3xz + 2yz + z^2 \). Find the greatest rate of increase of \( f \) at the point \( (1, -2, 1) \) and a unit vector pointing in the direction of greatest increase.

15. The figure below is a contour map of a function \( f(x, y) \). At the point \( (2, 2) \), sketch a unit vector in the direction of \( \nabla f(2, 2) \).
16. Find the equation of the tangent plane to \( z = xe^{-2y} \) at (1,0,1).

17. Approximate \( \Delta z \) for \( z = \ln(xy^2) \) as \((x,y)\) moves from \((4,-2)\) to \((2,-1)\) using the differential.

18. Find the largest and smallest value of \( f(x,y) = x - x^2 - y^2 \) if \( x \) and \( y \) are on the unit circle.
19. Find the critical points of $f(x, y) = x^4 + y^4 - 4x - 32y + 10$. Classify them using the Second derivative test.

20. Find the critical points of $f(x, y) = xye^{-x-y}$. Classify them using the Second derivative test.
21. Use Lagrange Multipliers to find the extreme values of \( f(x, y) = xy \) subject to the constraint \( 4x^2 + y^2 = 8 \).

22. Use Lagrange Multipliers to find the extreme values of \( f(x, y) = xe^y \) subject to the constraint \( x^2 + y^2 = 2 \).
MA 250 – Practice Problems for Mastery Exam 3

Review all examples from Notes, Homework, Ice sheets, and Quizzes. Know basic theorems and definitions. Make sure you can do all of those problems. You can also practice with the problems below. Note you can use a calculator on all problems including sketching polar curves. Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

Here are the 5 Mastery Concepts:

1. Double Integrals
   • Be able to set up and compute double integrals using both Cartesian and Polar Coordinates for general regions.
   • Be able to switch the order of integration for a double integral.
   • See problems 23-28

2. Triple Integrals
   • Be able to set up and compute triple integrals using Cartesian, Cylindrical, and Spherical Coordinates depending on the regions of integration.
   • Be able to switch the order of integration for a triple integral.
   • See problems 8-11, 21

3. Change of Variables

4. See problems 12, 13, and notes, ICE, and HW examples
   • Be able to set up an integral by using change of variables. You will not need to compute the integral, but you will need to set up the integral completely using the new variables.

5. The Fundamental Theorem of Line Integrals
   • Be able to prove a Vector Field is conservative, find a potential function, and use the potential function to calculate a line integral using the fundamental theorem of line integrals.
   • See problems 1, 2, 14, 20, 22

6. Line Integrals and Green’s Theorem
   • Be able to recognize when we can use Green’s Theorem and use it appropriately.
   • Be able to parameterize a curve so we can calculate the line integral without using the fundamental theorem of line integrals.
   • See Problems 3-7 and 15-19
1. Consider the vector field: \( \mathbf{F}(x, y, z) = yze^{xz} \mathbf{i} + e^{xz} \mathbf{j} + xye^{xz} \mathbf{k} \)

   a) Determine \( \text{Div} \mathbf{F} \)

   b) Determine \( \text{Curl} \mathbf{F} \)

   c) Determine whether \( \mathbf{F} \) is conservative

   d) Find a potential function for \( \mathbf{F} \)

2. Determine \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is a simple curve from (0,1,0) to (1,1,1) and \( \mathbf{F} = \nabla(3xyz + xe^z) \)
3. Evaluate $\int_C xy\,ds$ where $C$ is the circle with radius 2 on the oriented path from $(2,0)$ to $(0,2)$.

4. Evaluate $\int_C xy\,ds$ where $C$ is the circle with radius 2 on the oriented path from $(0,2)$ to $(2,0)$.

5. What is the relationship between $\int_C fds$ and $\int_{-C} fds$?

6. Evaluate $\int_C xy\,ds$ where $C$ is the line from $(2,0)$ to $(0,2)$. 
7. Evaluate $\int_C xy \, ds$ where $C$ is the line from $(0, 2)$ to $(2, 0)$.

8. $\iiint_V (1) \, dV$ where $V$ is the region in the first octant bounded by $y = 2x^2$ and $y + 4z = 8$.

9. Evaluate $\int_{-1}^{1} \int_{3x^2}^{4-x^2} \int_{0}^{6-z} dy \, dz \, dx$.

10. Use cylindrical coordinates to find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 25$ and below by $z = 4$. 
11. Find the volume of the region above the cone \( z = \sqrt{x^2 + y^2} \) and below the sphere \( x^2 + y^2 + z^2 = z \).

12. Find the Jacobian for the transformation \( x = r \sin t \) and \( y = r \cos t \).

13. Use a change of variables to rewrite the integral (you do not have to evaluate the integral).
\[
\iint_R \sqrt{\frac{x - y}{x + y + 1}} \, dA \quad \text{where R is the square with vertices (0, 0), (1, −1), (2, 0), (1, 1).}
\] Sketch the original region and the new region.

14. If \( \mathbf{F} = (x^2 - y^2)\mathbf{i} + 2xyz\mathbf{j} + z^2\mathbf{k} \)
   
   a) Find \( \text{div} \mathbf{F} \)
b) Find \( \text{curl} \mathbf{F} \)

c) Can we use the fundamental theorem of line integrals to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) for simple curve \( C \).

15. Compute \( \int_C (x^2 - y) \, dx + (y^2 + x) \, dy \) from \((0,1)\) to \((1,2)\) along the curve \( C : y = x^2 + 1 \).

16. A particle travels along the helix \( C \) given by \( \mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + 2t \mathbf{k} \) and is subject to the force \( \mathbf{F}(x, y, z) = \langle x, z, xy \rangle \). Find the total work done on the particle by the force for \( 0 \leq t \leq 3\pi \).

17. Evaluate \( \int_C \sqrt{1 + x^3} \, dx + xy \, dy \) where \( C \) is a positively oriented triangle with vertices \((0,0), (2,0), \) and \((0,3)\).
18. Use Green’s theorem to evaluate \[ \oint_{C} (3x - 4y)\,dx + (4x - 2y)\,dy \] where \( C \) is the path counter-clockwise around the ellipse \( x^2 + 4y^2 = 16 \) beginning and ending at \( (4,0) \).

19. Use Green’s theorem to evaluate \[ \oint_{C} (x^2y)\,dx + (2x)\,dy \] where \( C \) is the boundary of the triangle with vertices \((0,0)\), \((1,0)\), and \((1,1)\) oriented clockwise.

20. Know the equivalent requirements for the Fundamental theorem of calculus and know the fundamental theorem of calculus and terms like conservative vector fields, etc.

21. Swap the limits of integration for \[ \int_{0}^{2} \int_{0}^{4-2y} \int_{0}^{4-2y-z} \,dxdzdy \] to \[ dzdydx \]
22. Confirm that \( \mathbf{F}(x, y) = \langle 2e^y - ye^x, 2xe^y - e^x \rangle \) is conservative and find a potential function. Then what is the value of \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is a simple curve from \((1,1)\) to \((0,1)\).

23. Evaluate \( \iint_R \frac{1}{\sqrt{16 - x^2 - y^2}} \, dA \) where \( R = \{(x,y) : x^2 + y^2 \leq 4, x \geq 0, y \geq 0\} \)
24. Consider a plane given by the equation \( z = ax + by + c \), where \( a, b, c > 0 \). Find the volume of the solid lying under the plane and above the rectangle \( R = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\} \). Your answer may depend upon \( a, b, c \).

25. Set up and evaluate an iterated integral that represents the surface area of \( f(x, y) = 2x + \frac{2}{3}y^{\frac{3}{2}} \) over the triangle with vertices \((0, 0), (2, 0), \) and \((2, 2)\).
26. Change the order or integration for $\int_0^2 \int_{x^2}^{2x} f(x,y) \, dy \, dx$

27. Sketch the following region and write an iterated integral of a continuous function $f$ over $R = \{(x, y) : 0 \leq x \leq 4, x^2 \leq y \leq 16\}$

28. a) Evaluate $\iint_R xy \, dA$ where $R$ is bounded by $x = 0$, $y = 2x + 1$, and $y = -2x + 5$.

   b) Then change the limits of integration.
MA 250 – Practice Problems for Mastery Exam 4

Review all examples from Notes, Homework, Ice sheets, and Quizzes. Know basic theorems and definitions. Make sure you can do all of those problems. You can also practice with the problems below. Note you can use a calculator on all problems including sketching polar curves. Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

Here are the 3 Mastery Concepts:

1. Vector Fields
   • Know how to graph and recognize graphs of vector fields.
   • Be able to tell whether or not a vector field is conservative/path independent/gradient field
   • Look at Vector Field and Curl and Divergence Notes and HW

2. Surface Integrals
   • Be able to set up and compute surface integrals and flux integrals.

3. Stokes and Gauss’ Divergence Theorems
   • Be use Stokes’ and Divergence Theorems appropriately.
   • be able to rewrite integrals using these theorems.
1. Verify that the line integral and surface integral of Stokes’ Theorem are equal for the following vector fields:
   a) \( \mathbf{F} = \langle 0, -x, y \rangle \) where \( S \) is the upper half of the sphere \( x^2 + y^2 + z^2 = 4 \) and \( C \) is the circle \( x^2 + y^2 = 4 \) in the xy plane.

   b) \( \mathbf{F} = \langle x, y, z \rangle \) where \( S \) is the paraboloid \( z = 8 - x^2 - y^2 \) for \( 0 \leq z \leq 8 \) and \( C \) is the circle \( x^2 + y^2 = 8 \) in the xy plane.
2. Use Stokes’ Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = < -y, -x - z, y - x >$ where $C$ is the boundary of the part of the plane $z = 6 - y$ that lies in the cylinder $x^2 + y^2 = 16$.

3. Calculate $\iint_{\partial S} \mathbf{F} \cdot \mathbf{n} dS$ where $\mathbf{F} = < x^2, y - z^2, x >$ where $S$ is the solid $0 \leq y^2 + z^2 \leq 1$, $0 \leq x \leq 2$.

4. Calculate $\iint_{\partial S} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = < x^2, y^2, z^2 >$ where $S$ is the solid enclosed by $x + y + z = 4$, $x = 0$, $y = 0$, $z = 0$. 

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5. Evaluate the flux of $\mathbf{F} = < x, y, 3 >$ across the sphere $x^2 + y^2 + z^2 = 1$.

6. Evaluate $\int \int_{G} (2y^2 + z) dS$ where $G : z = x^2 - y^2$ for $0 \leq x^2 + y^2 \leq 1$.

7. Evaluate $\int \int_{G} y dS$ where $G : z = 4 - y^2$ for $0 \leq x \leq 3, 0 \leq y \leq 2$. 
8. Find a function $f$ satisfying $\nabla f = (2xy + y)i + (x^2 + x + \cos y)j$

9. Find a function $f$ satisfying $\nabla f = (yz - e^{-x})i + (xz + e^y)j + xyk$

10. Find the work done by $\mathbf{F} = y^2i + 2xyj$ in moving an object from (1, 1) to (3, 4)

11. Evaluate $\oint_C (e^{3x} + 2y)dx + (x^2 + \sin y)dy$ where $C$ is the rectangle with vertices (2, 1), (6, 1), (6, 4), (2, 4)
35  Homework

Please write your solutions on these homework pages and show enough of your work so that I can follow your thought process. This makes it easier for me to grade. Also please staple the homework together before you turn it in. Sometimes I have my stapler, but there is also a stapler in my office and at the front desk of the Department of Computer and Mathematical Sciences.

Follow the instructions for each question. If I can’t read your work or answer, you will receive little or no credit!
1. Evaluate $\int_0^8 xe^x \, dx$

2. Evaluate $\int \sqrt{3 - x^2} \, dx$
3. Evaluate \( \int \frac{x^2}{4 - x^2} \, dx \)

4. Evaluate \( \int \sin^3 x \sqrt{\cos x} \, dx \)
5. Find the equation(s) of the tangent line(s) to \(x = \sin t, \ y = \sin(t + \sin(t))\) at (0, 0). for \(0 < t < 2\pi\).

6. Determine \(\frac{d^2y}{dx^2}\) without eliminating the parameter for \(x = 3 - 2 \cos t, \ y = -1 + 5 \sin t\) for \(t \neq n\pi\) for natural number \(n\).
1. One way to accurately render three-dimensional objects on a computer screen involves using the dot and cross products. In order to determine how to shade a piece of a surface, we need to determine the angle at which rays from the light source hit the surface. If the light ray hits the surface straight on, then this piece of the surface will appear bright. On the other hand, if the light comes in on an angle, this piece of the surface will not appear as bright.

Suppose the light source is placed directly above the $xy$-plane, so the light rays come in parallel to the vector $<0,0,-1>$.

a) Consider the plane containing the points $(3,2,4)$, $(2,5,3)$, and $(1,2,6)$. Find two vectors that are in the plane, and then determine a vector orthogonal (normal) to the plane.

b) Find the angle between the light rays an the normal vector to the plane.
2. You are at the point \((-1, -3, -3)\) standing upright and facing the \(yz\) plane. You walk 2 units forward, turn left, and walk for another 2 units.

a) What is your final position?

b) From the point of view from your position (and where you are facing in the coordinate system), are you above, below, in front of, behind, to the right of, or to the left of the \(yz\)-plane?

c) Are you above, below, to the right of, or to the left of the \(xy\)-plane?

3. A woman walks due west on the deck of a ship at 3 mph. The ship is moving north at a speed of 22 mph. Find the speed of the woman relative to the surface of the water.

4. The Amazing Math Kitties, Eva and Archer passed Calculus II and are now in Calculus III. Today they are working on equations in 3-space. Eva says the graph is a plane, but Archer says the graph of the equation \(y = 3x + 2\) is a line that has a slope of 3 and a \(y\) intercept of 2. Who is correct? Are they both correct? Explain.
5. A wagon is pulled a distance of 100 meters along a horizontal path by a constant force of 50 N. The handle of the wagon is held at an angle of 30° above the horizontal. How much work is done?

6. A 100-meter dash is run on a track in the direction of the vector \( \vec{v} = 4\hat{i} + 3\hat{j} \). The wind velocity \( \vec{w} = 6\hat{i} + 4\hat{j} \) km/hr. The rules say that a legal wind speed measured in the direction of the dash must not exceed 5 km/hr. Will the race results be disqualified due to an illegal wind? [Hint: think projections...]

7. Now the fabulous Math Kitties are talking about vectors. Archer says that he thinks \(|u + v| = |u| + |v|\).
   a) Can you help Eva come up with an example in which Archer’s statement is true?

   b) Is Archer right in general? What should Eva say in response?
1. Eva and Archer, the amazing math kitties, are working on their homework. Unfortunately, Archer spilled milk on the one of the questions so all he can see is

\[(u \times v) \cdot v = \square.\]

Eva tells him he can still answer the question. Why is Eva correct and what is the answer to the question?

2. Archer wants to find the position function of a particle given that the velocity function is \(v(t) = 2t\mathbf{j} + \sin(t)\mathbf{k}\) and the initial position is \(\mathbf{r}(0) = \mathbf{i} + \mathbf{j}\). Archer says this means the constant of integration is \(<1, 1, 0>\), but Eva warns him to be careful about using the initial position properly. Help Archer find the correct position function.
3. Find the symmetric equations of the line passing through the points \((-1, 0, 5)\) and \((4, -3, 3)\).

4. Given the following curve: \(\mathbf{r}(t) = <2t, 4\sin(t), 4\cos(t)>\), answer the following:
   a) Find the unit tangent vector.
   b) Find the curvature \(\kappa\) of \(\mathbf{r}(t)\).
5. The exquisite math cats, Eva and Archer, are trying to determine the tangential and normal components of acceleration ($a_T$ and $a_N$) for $\mathbf{r}(t) = <t^3, t^2>$. Archer says to find $a_T$, he should dot $<3t^2, 2t>$ with $<6t, 2>$ Eva says that he is close but not quite right. What did Archer forget? Then find $a_T$ and $a_N$.

Turn over for the last problem.
6. Professor Keleher is traveling along the curve given by \( r(t) = <-2e^{3t}, 5\cos t, -3\sin(2t)> \). If the power thrusters are turned off, his ship flies off on a tangent line to \( r(t) \). He is almost out of power when he notices that a station on Octapa is open at the point with coordinates \((1.5, 5, 3.5)\). Quickly calculating his position, he turns off the thrusters at \( t = 0 \). Does he make it to the Octapa station? Show your work, and explain your answer in one complete sentence.
1. Consider the ellipse given by the vector function $r(t) = 6 \cos t \mathbf{i} + 3 \sin t \mathbf{j}$ and its graph below:

(a) Sketch the vectors $\mathbf{T}(\pi/2)$ (the unit tangent vector) and $\mathbf{N}(\pi/2)$ (the principle unit normal vector) at the point on the ellipse corresponding to $t = \pi/2$. Make sure to denote which is which. (You should not need to do any calculations.)

(b) By looking at the graph, decide at which two points on the ellipse the curvature is minimal. List the two points.

2. The following is a contour map for the function $f(x, y)$. Determine the signs of $f_x, f_y, f_{xx},$ and $f_{yy}$ at the point P. Justify your answers.
3. Consider \( f(x, y, z) = y^2 \sin(x^3 + z^2) \)

   (a) Determine \( f_x \) and \( f_z \).

   (b) Determine \( f_{xx} \) and \( f_{zx} \).

   (c) Archer is attempting to determine \( f_{xxxyyy} \). Eva says that he can easily justify that the answer is 0 without doing any computation. Help Eva explain why this is so.
4. Remember, you can use a calculator to help you.

(a) Sketch both $r = 1 + \cos \theta$ and $r = 1 - \cos \theta$ on the same axis.

(b) Find the area of the region that lies inside both $r = 1 + \cos \theta$ and $r = 1 - \cos \theta$.

5. Below is the Cartesian graph of $y = f(x)$ with $0 \leq x \leq 2\pi$. Using the same function $f$, sketch a graph in polar coordinates for $r = f(\theta)$.
6. Find the slope of the tangent line to \( r = \cos(2\theta) \) at \( \theta = \frac{\pi}{4} \).

7. The Amazing Math Kitties are working on evaluating limits in Calculus 3. First they are exploring \( \lim_{(x,y) \to (0,0)} \frac{xe^y - x}{y} \).

(a) Archer says that because he gets \( \frac{0}{0} \) when he plugs in 0 for \( x \) and \( y \), the limit is undefined. Eva reminds him sometimes we still have limits exist with this indeterminate form. In fact we could use L’Hopital’s Rule to show \( \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \). Use L’Hopital’s Rule to verify Eva’s example.

(b) Archer, now convinced, excitedly says that he will now use L’Hopital’s Rule for this limit. Eva says first he needs to separate the limit before he can use L’Hopital. Help Archer separate the limit into the product of two singular variable functions. Then use L’Hopital’s Rule to help Archer calculate the limit.
8. Now Eva and Archer are working on \( \lim_{(x,y) \to (0,0)} \frac{y}{\sqrt{x^2 - y^2}} \):

(a) Archer says that he will use L'Hopital's Rule for this limit since it worked in the last example. Eva shakes her head. Why does Eva not agree? Why does this method not work in this example?

(b) Archer says he will now try to evaluate this limit by trying the path \( x = 0, y = y \). "Careful!" Eva warns. "That is not a valid path we can use." Explain why this is not a valid path.

(c) Help Archer find at least two different paths that go to two different limits. What does this mean about our limit?

Turn over for last problem.
(d) Eva says that she uses Polar Coordinates to solve $\lim_{(x, y) \to (0, 0)} \frac{y}{\sqrt{x^2 - y^2}}$. Use Eva’s method to show this limit doesn’t exist.
35.5 MATH 25000 – HW 5 due Friday, 10/27

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can’t read your work or answer, you will receive little or no credit!

1. Designing safe boilers depends on knowing how steam behaves under certain changes in temperature and pressure. Steam tables, such as the one below, are published giving values of the function \( V = f(T, P) \) where \( V \) is the volume (in cubic feet) of one pound of steam at a temperature of \( T \) (in degrees Fahrenheit) and pressure \( P \) (in pounds per square inch).

<table>
<thead>
<tr>
<th>( T ) ( / ) ( P )</th>
<th>20</th>
<th>22</th>
<th>24</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>480</td>
<td>27.85</td>
<td>25.31</td>
<td>23.19</td>
<td>21.39</td>
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<tr>
<td>500</td>
<td>28.46</td>
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<td>520</td>
<td>29.06</td>
<td>26.41</td>
<td>24.20</td>
<td>22.33</td>
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<tr>
<td>540</td>
<td>29.66</td>
<td>26.95</td>
<td>24.70</td>
<td>22.79</td>
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</table>

a) Find the tangent plane to \( V = f(T, P) \) for \( T \) near 500°F and \( P \) near 24 lb/in\(^2\). [Hint: Use the tables to approximate your partial derivatives and recall the equation of our tangent plane is: \( V = V_T(T - T_0) + V_P(P - P_0) + f(T, P) \) ]

b) Use the tangent plane to estimate the volume of a pound of steam at a temperature of 505°F and \( P \) near 24.3 lb/in\(^2\).
2. Find the directional derivative of \( f(x, y) = \ln(xy) \) at \( \left( \frac{1}{2}, \frac{1}{4} \right) \) in the direction of \( <1, 1> \).

3. a) Find the maximum and minimum rates of change of \( f(x, y, z) = e^{xyz+1} \) at \( (0, -1, 1) \) and the directions in which they occur. Write your answer in the forms below.

We have a maximum rate of change of ________ in the direction of ________

We have a minimum rate of change of ________ in the direction of ________

b) Find the unit vector in the direction of maximum increase of \( f \) at \( (0, -1, 1) \).
4. The Super Math Kitties, Eva and Archer are exploring the gas equation for one mole of oxygen relates its pressure $P$ (in atmospheres), its temperature, $T$ (in K), and its volume, $V$ (in cubic decimeters):

$$T = 16.574 \frac{1}{V} - 0.52754 \frac{1}{V^2} + 0.3879P + 12.187VP.$$ 

a) What does $T_V$ represent? Archer say that the $T_V$ is the rate at which the temperature changes when pressure increases and the volume is kept constant. Is he right? If not, adjust his representation.

b) Help Archer find $T_V$ when the volume is 25 dm$^3$ and the pressure is 1 atmosphere.

c) Now determine $T_P$ when the volume is 25 dm$^3$ and the pressure is 1 atmosphere.

d) What does the total differential of $T$, $dT$, represent?
e) Find an expression for the **total differential** $dT$ in terms of the differentials $dV$ and $dP$ when the volume is $25 \text{ dm}^3$ and the pressure is 1 atmosphere. Archer gives you a hint that $dT = <T_v, T_P> \cdot <dV, dP>$ and Eva reminds you that your answer should have the terms $dT, dV, \text{ and } dP$ in it and to use your answers from b and c.

f) Eva now wants to use your answer above to estimate how much the volume would have to change if the pressure increased by 0.1 atmosphere and the temperature remained constant. Archer reminds you that we are approximating $\Delta V$ using $dV$.

g) Help Archer write your answer in the previous question in the form of “the volume is ______________ at a rate of ________ cubic decimeters”.

5. Suppose that the price $P$ (in dollars), to purchase a used car is a function of $C$, its original cost (also in dollars), and its age $A$ (in years). So $P = f(C,A)$. Is the sign of $\frac{\partial P}{\partial C}$ positive, negative, or zero. Explain.
6. Wheat production in a given year, \( W \), depends on the average temperature \( T \) and the annual rainfall \( R \). Scientists estimate that the average temperature is rising at a rate of 0.14\(^\circ\)/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that, at current production levels, \( \frac{\partial W}{\partial T} = -2 \) and \( \frac{\partial W}{\partial R} = 8 \). Estimate the current rate of change of wheat production \( \frac{dW}{dt} \).

7. Given \( 3x^2z + y^3 - xy^2z^3 = 0 \), use Calculus III techniques to determine the following:

   a) \( \frac{dy}{dx} \).

   b) \( \frac{\partial z}{\partial x} \).
Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can’t read your work or answer, you will receive little or no credit!

1. Suppose you need to cut a beam with maximal rectangular cross section from an elliptical log with semi-axes of lengths 2 ft and 1 foot. Use Lagrange multipliers to find the maximal cross-sectional area of such a beam. Include correct units in your answer. (If coordinate axes are set up so the center of the log is at the origin, the log is then bounded by the ellipse $x^2 + 4y^2 = 4$. See the figure below.)
2. Find and classify all six critical points of \( f(x, y) = 3x - x^3 - 2y^2 + y^4 \).

Write your answers in the chart below

<table>
<thead>
<tr>
<th>Critical Point</th>
<th>Value of D</th>
<th>sign of ( f_{xx} )</th>
<th>Conclusion</th>
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![Contour plot of the function](image)
3. Eva and Archer want to calculate $\int_{-2}^{2} \int_{0}^{2} |3x^3y^2| \, dy \, dx$.

a) Archer is a little confused by the absolute values, he uses his absolute value properties to rewrite $|3x^3y^2|$ as $3|x^3| \cdot |y^2|$. Eva says that this is an excellent start since it allows them to separate the integral into $\int_{-2}^{2} \int_{0}^{2} |3x^3y^2| \, dy \, dx = \int_{-2}^{2} |x^3| \, dx \cdot \int_{0}^{2} 3|y^2| \, dy$. Is Eva correct? Why or why not?

b) Archer is still a little stuck, but Eva says he can use symmetry to help eliminate the absolute values. In particular, help Archer rewrite $\int_{-2}^{2} |x^3| \, dx$ without absolute values. Then evaluate the integral.

c) Now help Archer rewrite $\int_{0}^{2} |y^2| \, dy$ without absolute values. Then evaluate the integral.

d) Use your answers from part b and c to calculate $\int_{-2}^{2} \int_{0}^{2} |3x^3y^2| \, dy \, dx$. 
4. Set up, but **do not** evaluate an integral that represents the volume of a tetrahedron bounded by the coordinate planes and the plane \( z = 6 - 2x - 3y \).

5. a) Set up the integral \( \int \int_S (x^2 - xy) \, dA \), where \( S \) is the region between \( y = x \) and \( y = x^2 \) using \( dy \, dx \).

b) Set up the integral using \( dx \, dy \).
6. Find the coordinates of the center of mass of the following plane region: the quarter disk in the first quadrant bounded by $x^2 + y^2 = 4$ with density $1 + x^2 + y^2$. Recall the center of mass is given by $(\bar{x}, \bar{y})$ where 
$$
\bar{x} = \frac{\iint_R x \delta(x,y) dA}{\iint_R \delta(x,y) dA}
$$
and 
$$
\bar{y} = \frac{\iint_R y \delta(x,y) dA}{\iint_R \delta(x,y) dA}.
$$
Archer says to use symmetry!
35.7 MATH 25000 – HW 7 due Monday: 11/20

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can’t read your work or answer, you will receive little or no credit!

1. Consider the double integral \( \iiint_S (x^2 + y^2 + z^2) \, dV \) where S is the following solid, which is the piece of the unit sphere in the first octant.

(a) Set up the triple integral as an iterated integral in rectangular coordinates. DO NOT EVALUATE!

(b) Set up the triple integral as an iterated integral in cylindrical coordinates. DO NOT EVALUATE!

(c) Set up the triple integral as an iterated integral in spherical coordinates. DO NOT EVALUATE!
2. Change the order of the integration for \( \int_0^2 \int_0^{4-2y} \int_0^{4-2y-z} f(x,y,z) \, dx \, dz \, dy \) to \( dz \, dy \, dx \). AND sketch the region of the integration.

3. Archer and Eva are trying to sketch the solid region of integration for the triple integral 
\( I = \int_0^1 \int_0^{1-y} \int_0^{1-x} 2x \, dx \, dy \, dz \).
Archer has graphed the region below. Is he correct? If not, fix his graph. Eva says it helps her to draw the footprints in each of the coordinate planes (xy, xz, and yz.)
4. a) Set up a **double** integral in polar coordinates that represents the volume of the solid region enclosed by the paraboloid $z = x^2 + y^2$ and the plane $r = 4$.

b) Set up a **triple** integral that represents the volume of the solid region enclosed by the paraboloid $z = x^2 + y^2$ and the plane $r = 4$.

5. **Evaluate** the integral $\iiint_D \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} dV$ where $D$ is the region between the spheres of radius 1 and 2 centered at the origin. Eva says to use a good coordinate transformation!
6. a) Evaluate $\iint_R \sqrt{y^2 - x^2} \, dA$, where $R$ is the diamond bounded by $y - x = 0, y - x = 2, y + x = 0,$ and $y + x = 2$. Eva notices that the bounds can be used to guide us to a nice $u$ and $v$. Archer agrees and says that letting $u = y - x$ will also help with the integrand since $y^2 - x^2 = (y - x)(y + x)$.

b) Sketch the original and new regions of integration.
1. a) Sketch each vector field below and use the sketch to decide whether the divergence is positive, negative, or zero at the point (1, 1). Recall, the divergence at a point calculates the net outflow per unit of volume. The divergence is positive if the net outflow is positive. It is negative if the net outflow is negative (we have inflow). When the divergence is zero, the net outflow is zero and we say our vector field is incompressible.

b) Check your answer by calculating the divergence of $\mathbf{F}$.

c) Determine whether the vector field is conservative or not.

\begin{align*}
a) \quad & \mathbf{F}(x, y) = -y \mathbf{j} \\
b) \quad & \mathbf{F}(x, y) = < y, y + 2 > \\
c) \quad & \mathbf{F}(x, y) = \mathbf{j} - \mathbf{i}
\end{align*}
2. Water is an essentially incompressible fluid, that is the divergence of a velocity field representing the flow of water is 0. Determine which of the following vector fields could represent the velocity vector field for water flowing.
   a) \( \mathbf{F}(x, y, z) = xy \mathbf{i} + xz \mathbf{j} - yz \mathbf{k} \)

   b) \( \mathbf{F}(x, y) = < xy, z \sin x, e^{xy} > \)

3. Evaluate \( \int_C (2x - 3y)ds \) where \( C \) is the line segment from \((-1, 0)\) to \((0, 1)\).

4. Evaluate \( \iint_G xy dS \) if \( G \) is the plane \( z = 2 - x - y \) in the first octant.
5. Consider the vector field, $\mathbf{F}(x, y) = <yz, xz, (xy + 2z)>$

a) Calculate the curl of $\mathbf{F}$ and show that $\mathbf{F}$ is a conservative vector field.

b) Find a potential function $f$ for $\mathbf{F}$.

c) Use your answer from part b and the Fundamental Theorem of Line Integrals to evaluate $\int_{C} \mathbf{F} \cdot d\mathbf{r}$ where $C$ is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$. 
6. Suppose we have the force field \( \mathbf{F} = \langle -y, x, z \rangle \). Find the work required to move an object on the helix \( \mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, \frac{t}{2\pi} \rangle \) for \( 0 \leq t \leq 2\pi \).

Can we use the fundamental theorem or line integrals? Why or why not?

7. Use the Divergence Theorem to compute the net outward flux of the vector field \( \mathbf{F} = \langle x, -2y, 3z \rangle \) across the surface of the sphere \( x^2 + y^2 + z^2 = 6 \).
8. The Amazing Math Kitties, Eva and Archer are trying to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = \langle y, xz, -y \rangle$ where $C$ is the ellipse $x^2 + \frac{y^2}{4} = 1$ with counterclockwise orientation in the plane $z = 1$. Archer says that he will parameterize the ellipse using $x = \cos t$, $y = 2\sin t$, $z = 1$, Eva says she can use Stoke’s Theorem. Which kitty is correct? Are they both correct? If so, calculate the integral using both methods.
This is your last HW for the year!

1. Please go on blackboard and complete the evaluation for this Calc III class. Please give thoughtful, constructive feedback which will help me improve the course. For example, saying “You suck” or “You are great” doesn’t provide much feedback for me. Saying “You suck because...” or “You are great because...” Also, remember for everything you like about the course, there is at least one other person who dislikes it, so please let me know what you would like to be kept the same about the course.

Check one:

☐ I completed the evaluation.
☐ I have not completed the evaluation.

2. Create a Meme about this course. It can be something about the topic we covered this semester, but it should relate in some way to this course. On the last day of school, we will share all of the memes and vote for the best one. [Note: You can use a meme that already exists if it relates to this course, but it is more fun to create your own.] I will create an assignment in blackboard in which you can upload your Meme or you can print it and attach to this paper.

3. Create a “good” Archer answer relating to something from what we learned in class this semester. So this should be an incorrect answer (but not a trivial incorrect answer) that demonstrates a subtle misconception about a concept or topic in this course. Then write what Eva should say in order to help correct his mistake and explain what the misconception is.