MATH 36100: Real Analysis II Lecture Notes

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Contents

0 Syllabus Crib Notes

The full syllabus is posted in Blackboard with sample proof rubrics. Here are some highlights from the syllabus:

0.1 Office Hours

Please come to my office hours! Helping you with the material is the best part of my job! I have 5 weekly office hours which I will hold. My office is in **AS-124-A**. Remember if none of these times work, send me an email and we can schedule another time to meet. I can also answer questions through email! This semester my office hours will be:

Mondays: 10:30-11:00
Tuesdays: 11:00-12:00 and 1:00-2:00
Wednesdays: 10:30-11:00
Thursdays: 1:00-2:00
Fridays: 2:00-3:00
Or By Appointment!

Note: Sometimes I have meetings or class that goes right up to my office hours, so if I am not there, please wait a few minutes. Also sometimes I have unexpected meetings that get scheduled during my office hours. If this happens, I will do my best to let you know as soon as possible and I usually hold replacement office hours.

Help: Don't wait to get help. Visit me during my office hours, use the discussion forum in Blackboard, go to the Math Study Tables, find a study partner, get a tutor!

Dr. Harsy's web page: For information on undergraduate research opportunities, about the Lewis Math Major, or about the process to get a Dr. Harsy letter of recommendation, please visit my website: `http://www.cs.lewisu.edu/~harsyram`.

Blackboard: Check the Blackboard website regularly (at least twice a week) during the semester for important information, announcements, and resources. It is also where you will find the course discussion board. Also, check your Lewis email account every day. I will use email as my primary method of communication outside of office hours.

0.2 Grades

Category	Percentage
Homework	60
Productive Engagement	20
Portfolio	20

Dr. Harsy reserves the right to change the percentages of these portions.

Productive Engagement: In order to achieve the maximum points for this portion of your grade you must actively Present, Facilitate, and Participate in class activities, present problems and proofs from homework and ICE sheets, and participate in class discussions.

Proof Portfolio: You will keep a Proof Portfolio written in a LaTeXfile. This will be an end of the year project in which I would like you to write up your top/favorite 5 proofs from the class. After each proof, give an explanation about why you chose this proof. These portfolios will be due the **second to last** day of class and we will go over your choices along with your memes on the last day of class. If you would like a template for this portfolio, let me know and I will put one together for it!

Team Homework: We will work through guided notes and in-class exercises. Each week a subset of these problems will be assigned as a Team Homework assignment which you can work on with a group of no more than 4 people. You will have one chance to revise this homework and for each assignment, you should include

- A summary of how you worked on this homework as a group and
- A signed statement stating that everyone contributed to this product.

The homework should be submitted as a team and should be written in LaTeXunless the problem requires pictures or there is a good reason to not Tex it up.

0.3 Expectations and Norms for the Course

1. I will embrace challenges because they help me learn.
2. I will not be afraid of making mistakes and taking risks because they provide learning opportunities.
3. I will be respectful of the diversity in the room.
4. I will be a mindful contributor and work as a team during classroom activities.
5. I will have a positive attitude about this class because my attitude is something I can control.
6. I will be appreciative of the effort others put forth during this semester.
7. I understand that assessment opportunities give me a chance to demonstrate my growth and learning.
8. I will minimize distractions during class.
9. I will help to create an inclusive learning community.

0.4 About the Course

"If intellectual curiosity, professional pride, and ambition are the dominant incentives to research, then assuredly no one has a fairer chance of gratifying them than a mathematician. His subject is the most curious of all-there is none in which truth plays such odd pranks. It has the most elaborate and the most fascinating technique, and gives unrivaled openings for the display of sheer professional skill. Finally, as history proves abundantly, mathematical achievement, whatever its intrinsic worth, is the most enduring of all..."

-G.H. Hardy

Thanks for taking Real Analysis I with me! Real Analysis is one of my favorite courses to teach. In fact, it was my favorite mathematics course I took as an undergraduate. You may be wondering, "What exactly is Real Analysis?"

Analysis is one of the principle areas in mathematics. It provides the theoretical underpinnings of the calculus you know and love. In your calculus courses, you gained an intuition about limits, continuity, differentiability, and integration. Real Analysis is the formalization of everything we learned in Calculus. This enables you to make use of the examples and intuition from your calculus courses which may help you with your proofs. Throughout the course, we will be formally proving and exploring the inner workings of the Real Number Line (hence the name *Real* Analysis). But Real Analysis is more than just proving calculus, and I think Dr. Carol Schumacher of Kenyan College describes it extremely well by when she calls *Analysis* the "Mathematics of Closeness." At its core, this is what Real Analysis is above. When you think about the derivatives and integration, remember we talk about taking small changes, Δx whether it be a $\frac{\Delta y}{\Delta x}$ or a partition for our Riemann Sums. Our job in Real Analysis is to understand how to formally describe closeness and the process of getting "closer and closer" (limits).

Recall, Real Analysis I started with very abstract concepts and became more concrete as the semester goes on. Remember, the hardest part of the class is at the beginning! We started by talking about bounds of real numbers which allows us to prove that there is in fact a unique limit we want to reach. We then explored sequences which we will use to get as close as we can to these numbers/bounds. Next we discussed closeness in a function setting along with continuity. We needed continuity later for our integration and special derivative theorems. We then revisited and use sequences and functions to discuss rate of change (derivatives) and optimization. We wanted to finish with Riemann Sums and the beautiful Fundamental Theorem of Calculus.

I really debated with what to do for Real Analysis II and I have decided to take a more abstract and general approach to the concepts in Real Analysis I. Some Analysis courses actually do this from the beginning, but at times you will be able to use your intuition from Real Analysis I to help you with the concepts here. Sometimes you will prove more general versions of the proofs from Real Analysis I. If we have time, we may add some point-set topology (maybe).

I hope you will enjoy this semester, learn a lot, and feel challenged by the material (in a good way)! Please make use of my office hours and plan to work hard in this class. My classes have a high work load (as all math classes usually do!), so make sure you **stay on top of your assignments and get help early**. Remember you can also email me questions if you can't make my office hours or make an appointment outside of office hours for help. When I am at Lewis, I usually keep the door open and feel free to pop in at any time. If I have something especially pressing, I may ask you to come back at a different time, but in general, I am usually available. I have worked hard to create this course packet for you, but it is still a work in progress. Please be understanding of the typos I have not caught, and politely bring them to my attention so I can fix them for the next time I teach this course. I look forward to meeting you and guiding you through the wonderful course that is Real Analysis Part 2 Electric Boogaloo.

Cheers,
Dr. H

Acknowledgments: No math teacher is who she is without a little help. I would like to thank my own undergraduate professors from Taylor University: Dr. Ken Constantine, Dr. Matt Delong, and Dr. Jeremy Case for their wonderful example and ideas for structuring excellent learning environments. I also want to thank Dr. Annalisa Crannell, Dr. Tom Clark, Dr. Alyssa Hoofnagel, Dr. Alden Gassert, Dr. Francis Su, Dr. Brian Katz, and Dr. Christian Millichap for sharing some of their resources from their own courses. *And finally, I would like to thank you and all the other students for making this job worthwhile and for all the suggestions and encouragement you have given me over the years to improve.*

1 Introduction

1.1 Setting the Stage

Welcome to the first day of Real Analysis II, before we dive in, I have a small activity for us to set the stage for the method of teaching in this course.

- First, form into groups of size 2-3.
- Group members should introduce themselves.
- For each of the questions below, I would like you to:
  1. Think about a possible answer on your own.
  2. Discuss your answers with the rest of your group.
  3. Share a summary of each groups discussion (each person will present at least one question.)

Question 1: What are the goals of a university education?

Question 2: How does a person learn something new?

Question 3: What do you reasonably expect to remember from your courses in 20 years?

Question 4: What is the value of making mistakes in the learning process?

Question 5: How do we create a safe environment where risk taking is encouraged and productive failure is valued?

"*Any creative endeavor is built on the ash heap of failure.*"

-Michael Starbird[1]

[1] University of Texas at Austin Mathematics Professor

1.2 Quick Review

Work with your group to come up with a quick recap of what you learned in Real Analysis. Perhaps make a list of major concepts and see if you can remember their definitions.

1.3 General Review

1. Is $\emptyset \in \mathbb{Q}$? Explain why or why not.

2. Is $0 \in \emptyset$? Explain why or why not.

3. Explain what is wrong with the following notations: $\mathbb{N} \in \mathbb{Q}$ and $1 \subset \mathbb{N}$.

4. Write in words what the following set means: $A = \{x \in \mathbb{Q} : x^2 < 3\}$.
   Dr. Harsy notation comment: I sometimes use "|" instead of ":" so in our class $A := \{x \in \mathbb{Q} : x^2 < 3\} = \{x \in \mathbb{Q} | x^2 < 3\}$

5. The **symmetric difference of sets A and B**, denoted $A \Delta B$ is a set of whose elements belongs to A but not to B, or belongs to B but not to A. Draw a picture representing what $A \Delta B$ represents for arbitrary sets A and B.

6. How do you show two sets are equal?

7. What are the 3 properties of an equivalence relation?

8. Explain why even though "Every integer is less than some prime number." is equivalent to "For each integer n there is a prime number p which is greater than n," the latter is better to use when converting to mathematical symbolic notation.

9. Give an example which shows that the following statement is false: $(\forall n \in \mathbb{N})(\exists p \in P)$ such that $(\forall m \in \mathbb{N})(nm < p)$.

10. Review DeMorgan's Law (note $\cap$ means "&"): $\neg(A \cap B) \equiv$

11. $a \implies b \equiv$ __________ $\equiv$ __________

1.3.1 Review of Sups and Infs (Supremum and Infimums)

Definition 1.1. *Let $A \subset \mathbb{R}$, and $s \in \mathbb{R}$, then s is an **upper bound** for A if $s \geq a \; \forall a \in A$*

Definition 1.2. *Let $A \subset \mathbb{R}$, and $s \in \mathbb{R}$, then s is the **LEAST upper bound** or **supremum** of A if*

1. *$s \geq a \; \forall a \in A$ (that is, s is an ____________________)*
2. *If $\tilde{s} < l$, then $\exists a \in A$ s.t. $a > \tilde{s}$.*

Definition 1.3. *Let $A \subset \mathbb{R}$, and $l \in \mathbb{R}$, then l is a **lower bound** for A if $l \leq a \; \forall a \in A$*

Definition 1.4. *Let $A \subset \mathbb{R}$, and $l \in \mathbb{R}$, then l is the **GREATEST lower bound** or **infimum** of S if*

1. *l is a lower bound AND*
2. *If $\tilde{l} > l$, then $\exists a \in A$ s.t. $a < \tilde{l}$.*

Helpful lemma:

Lemma 1.1. *Let $s \in \mathbb{R}$ be an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.*

Proof:

Example 1.1. *What would be the corresponding lemma for an infimum?*

Example 1.2. *For each set below, determine any min, maxs, sups, or infs.*

a) $[-4, 1)$

b) $\{2, 7\}$

c)$\{r \in \mathbb{Q} : r^2 < 1\}$

d) $\{n + \dfrac{(-1)^n}{n} : n \in \mathbb{N}\}$

e) $[0, 1] \cup [2, 7]$

f) $\{0\}$

g) $\{x \in \mathbb{R} : x < 0\}$

h) $\{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime } \}$

i) $\displaystyle\bigcap_{n=1}^{\infty}(1-\frac{1}{n}, 1+\frac{1}{n})$

j) $\displaystyle\bigcup_{n=1}^{\infty}(1-\frac{1}{n}, 1+\frac{1}{n})$

Example 1.3. *Which of the following are unique: an upper bound, a max, a sup, a lower bound, a min, an inf?*

2 Metric Spaces

Remember Analysis is the "study of closeness," so it is important to define what we mean by points or mathematical elements being close to one another by introducing a notion of *distance*. To do this we introduce a mathematical structure called **Metric Spaces.**

Definition 2.1. ***Metric Space:*** *We say a nonempty set X with associated function $d : X \times X \to \mathbb{R}$ is a **metric space** if the following conditions are satisfied:*

1. $\forall x, y \in X$, $d(x, y) \geq 0$
2. $\forall x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. $\forall x, y \in X$, $d(x, y) = d(y, x)$
4. $\forall x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$

If all of the conditions are satisfied, d is called a **distance function** or **metric**. The elements of a metric space are called **elements.**

Example 2.1. *In your own words, give a rationale for the properties that were required of this **metric** (distance function).*

Example 2.2. *What is the main metric for $\mathbb{R}^n$?*
$d(\boldsymbol{x}, \boldsymbol{y}) =$

Example 2.3. *Discuss why each axiom is true for $\mathbb{R}^n$ with the usual distance used for $\mathbb{R}^n$.*

Example 2.4. *Let X be any non empty space and define d in the following way:*

$$d(x,y) = \begin{cases} 0, & x = y \\ 1 & x \neq y \end{cases}$$

Is (X, d) a metric space? Justify or prove your answer.

Example 2.5. *Consider the space $\mathbb{R}^2$ and define d in the following way:*

$d(x,y) = d((x_1, y_1), (x_2, y_2)) = |x_1 x_2 + y_1 y_2|$

Is $(\mathbb{R}^2, d)$ a metric space? Justify or prove your answer.

Example 2.6. *Consider the space* $\mathbb{R}^2$ *and define* d *in the following way:*
$d(x, y) = d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$
Is $(\mathbb{R}^2, d)$ *a metric space? Justify or prove your answer. What if we extended this to* $\mathbb{R}^n$*?*

Example 2.7. *Let* $C[0, 1]$ *be the set of continuous functions on the closed interval* $[0, 1]$ *(That is functions of the form* $f : [0, 1] \to \mathbb{R}$*). Decide which of the following are metrics on* $C[0, 1]$*. Hint: it may be helpful to practice first and find the distance between* $f(x) = x^2$ *and* $g(x) = x$ *on* $[0, 1]$*.*

1. $d(f, g) = \max\{|f(x) - g(x)| : x, y \in [0, 1]\}$

2. $d(f, g) = |f(1) - g(1)|$

3. $d(f, g) = \int |f - g|$

Both of the following problems should be completed for homework:

Example 2.8. *Consider the metric space* $\mathbb{R}^2$*, but consider the metric given by*

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2|, & x_1 = x_2 \\ 1 + |y_1 - y_2| & x_1 \neq x_2 \end{cases}$$

Prove d *is a metric space on* $\mathbb{R}^2$

Example 2.9. *For* $p, q \in [0, \frac{\pi}{2})$ *let* $d_s(p, q) = \sin |p - q|$*. Use your calculus talent to decide whether* d_s *is a metric.*

Example 2.10. *Let $\mathcal{C}[a,b]$ denote the set of continuous functions $f : [a,b] \to \mathbb{R}$ and let $||f|| = \sup_{x \in [a,b]} |f(x)|$ and define $\rho(f,g) = ||f-g||$. Prove ρ is a metric on $\mathcal{C}[a,b]$.*

3 Introduction to Topology

3.1 Cardinality

In much of what follows below, we will consider the sets of things. We will not necessarily require that these are sets of real numbers. In particular we will look at the size of sets

Definition 3.1. *Let $f : A \to B$ be a function between two sets. Then f is called* ***one-to-one*** *if unique inputs map to unique outputs: if $a_1, a_2 \in A$ and $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$. f is called* ***onto*** *if every element of B has an element of A that maps to it: for every $b \in B$ there exists an $a \in A$ such that $f(a) = b$.*

Definition 3.2. *Two sets A and B have the same* ***cardinality*** *if there exists a function $f : A \to B$ that is one-to-one and onto (such a function is also called a* ***bijection****). In this case, we write $A \sim B$.*

Definition 3.3. *We say that a set A is* ***countable*** *if $\mathbb{N} \sim A$. If A is neither finite nor countable, then we say that A is* ***uncountable****.*

Example 3.1 (T/F)**.** *If $A = \{1, 2, 3\}$ and $B = \left\{e, \pi, \sqrt{2}\right\}$, then $A \sim B$.*

Example 3.2 (T/F)**.** *If $A = \{1, 2, 3\}$ and $C = \{x \in \mathbb{R} : (x^2 - 1)(x^2 - 4) = 0\}$, then $A \sim C$.*

Example 3.3 (T/F). *The even integers* $2\mathbb{Z}$ *have the same cardinality as the integers; that is,* $2\mathbb{Z} \sim \mathbb{Z}$.

Example 3.4 (T/F). $\mathbb{Z} \sim \mathbb{N}$*; that is, the integers are countable.*

Example 3.5 (T/F). *Let* P *denote the set of all prime numbers and let* $A_n = \{1, 2, \ldots, n\}$. *Then there exists an* $n \in \mathbb{N}$ *such that* $P \sim A_n$.

Example 3.6 (T/F). *If* $D = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) = \left\{x \in \mathbb{R} : \frac{-\pi}{2} < x < \frac{\pi}{2}\right\}$, *then* $\mathbb{R} \sim D$.

Example 3.7. *(a) Make a table of all positive rational numbers so that each fraction* $\frac{p}{q}$ *appears in the pth column and the qth row. (Okay, just go out as far as* $p, q = 5$*).*

(b) Cross out duplicates that are not in lowest terms.

(c) Turn your table $45°$ *clockwise, so that* $\frac{1}{1}$ *is in the top "row". There should be two numbers in the next row, and more numbers as you move further down. Reading this twisted table, list in order the first dozen numbers you encounter.*

Example 3.8 (T/F). $\mathbb{Q}$ *is countable.*

Example 3.9 (T/F). $\mathbb{R}$ *is countable/uncountable.*

3.2 Open and Closed Sets

3.2.1 Open Sets

The seemingly simple definitions we'll encounter in this section are fundamental to two important areas of mathematics: analysis and topology. It's going to be extremely useful to have a strong intuitive notion of what "open" and "closed" mean, and to have lots and lots of examples. For reasons you might discover later, topologists love open sets most of all. Analysts love closed sets most of all.

Definition 3.4. *An* ϵ***-ball*** *around the point* $a \in \mathbb{R}$ *is the set* $B_\epsilon(a) = \{x \in \mathbb{R} : d(x,a) < \epsilon\}$.

Actually, some books use the symbol "V_ϵ" or "U_ϵ," and uses the word **neighborhood**. Both "ball" and "neighborhood" are common words for this concept, but it's easier to spell "ball."

Definition 3.5. *An* ***open ball*** *with center* $\mathbf{p}$ *and radius* $r > 0$ *is given by* $B(p,r) = \{x \in E | d(p,x) < r\}$

Definition 3.6. *A set* $O \subset E$ *is* **open** *in* E *if and only if for every point* $p \in O$*, there is some* $r > 0$ *so that* $B(p,r) \subset O$*. That is, any* $q \in O$ *such that* $d(p,q) < r \implies q \in O$.

Note another way to define this can be $O \subset E$ is **open** in E if $\forall p \in O$, $\exists \epsilon > 0$ such that $B_\epsilon(p) \subset 0$.

Example 3.10. *Basically a set,* S*, is open if for any point* $p \in S$*, you can draw a little "ball" around it so that the ball stays in* S *Draw a picture that demonstrates the definition of an open set.*

Example 3.11. *Given* $\mathbb{R}$ *with the metric* $d(x,y) = |x-y|$*,*

1. *Is the set* $(0,1]$ *open?*

2. Is $(0, \infty)$ *open?*

3. Is $\mathbb{Q}$ *open?*

Example 3.12. *What types of intervals in* $\mathbb{R}$ *are open?*

Example 3.13. *Is the rectangle* $R = (a, b) \times (c, d) = \{(x, y) | a < x < b, c < y < d\}$ *open in* $\mathbb{R}^2$*?*

Example 3.14. *Given a metric space E with the discrete metric. Which sets are open?*
$d(x,y) = \begin{cases} 0, & x = y \\ 1 & x \neq y \end{cases}$

Theorem 3.1. *Any open ball, $B_\epsilon(p)$ (or $B(p,r)$) is an open set.*

Proof:

Example 3.15. *Consider the following subsets of* $\mathbb{R}$*.*

(a) *Is the interval* $(2,4) \cup (7,9)$ *open?*

(b) *Is the interval* $(2,4) \cap (7,9)$ *open?*

(c) *Is the interval* $[2,4]$ *open?*

(d) *Is the interval* $[2,4)$ *open?*

(e) Is $\mathbb{R} - \{5\}$ *open? (That's the real line with the point "5" removed).*

(f) Is the empty set open?

(g) Is $\mathbb{R}$ *open?*

(h) Is $\{5\}$ *open?*

(i) Is $\mathbb{Z}$ *open?*

Example 3.16. *Name a non-empty, bounded, open set that is not of the form* (a, b).

Example 3.17. *[T/F] The union of two open sets is an open set.*

Example 3.18. *[T/F] The union of finitely many open sets is an open set.*

Example 3.19. *[T/F] The union of countably many open sets is an open set.*

Example 3.20. *[T/F] The intersection of two open sets is an open set.*

Example 3.21. *[T/F] The intersection of finitely many open sets is an open set.*

Example 3.22. *[T/F] The intersection of countably many open sets is an open set.*

3.2.2 Closed Sets

Definition 3.7. *A* ***closed ball*** *with center* $\boldsymbol{p}$ *and radius* $r > 0$ *is given by*
$\bar{B}(p, r) = \{x \in E | d(p, x) \leq r\}$

Definition 3.8. *The* ***complement*** *of a set* $A \subseteq E$ *is the set* $\{x \in E | x \notin A\}$*(the set of elements not in A) and is denoted as* A^c*,* A'*, or* $E \backslash A$*.*

Definition 3.9. *A set* U *is* **closed** *if it is the complement of an open set.*

Example 3.23. *Given* $\mathbb{R}$ *with the metric* $d(x, y) = |x - y|$*, determine which of the following sets are closed.*
a) $(0, 1]$

b) $[0, \infty)$

c) $\{2, 7, 9\}$

Example 3.24. *Is* $\mathbb{Q}$ *closed?*

Theorem 3.2. *Any closed ball, $\bar{B}(p, r)$ is a closed set.*

Example 3.25. *Consider the following subsets of* $\mathbb{R}$.

a) Is the empty set closed?

b) Is $\mathbb{R}$ *closed?*

c) Is $\mathbb{R} - \{5\}$ *closed?*

d) Is $\{5\}$ *closed?*

e) Is $\mathbb{Z}$ closed?

f) Is the interval $(2, 4)$ closed?

g) Is the interval $[2, 4] \cap [5, 6]$ closed?

h) Is the interval $[2, 4] \cup [5, 6]$ closed?

i) Is the interval $[2, 4) \cup [5, 6]$ *closed?*

Example 3.26. *Which subsets of* E *with the discrete metric are closed?*

Example 3.27. *Name a non-empty, bounded, closed set that is not of the form* $[a, b]$.

Definition 3.10. *Subsets of a metric space that are* ***both*** *closed and open are called* ***clopen****set.*

Summary: A subset of a metric space can be _________, _________, _________, or _________.

Example 3.28. *There are only 2 clopen subsets of* $\mathbb{R}$ *with the Euclidean distance. What are they?*

Example 3.29. *[T/F] The union of two closed sets is a closed set.*
Hint: Use ***DeMorgan's Law:*** $(A \cup B)^C = A^c \cap B^c$ *and* $(A \cap B)^C = A^c \cup B^c$.

Example 3.30. *[T/F] The union of finitely many closed sets is a closed set.*

Example 3.31. *[T/F] The union of countably many closed sets is a closed set.*

Example 3.32. *[T/F] The intersection of two closed sets is a closed set.*

Example 3.33. *[T/F] The intersection of finitely many closed sets is a closed set.*

Example 3.34. *[T/F] The intersection of countably many closed sets is a closed set.*

Timeout for Nasty Math Pun:
Why are sets like doors?

Why are sets NOT like doors?

Theorem 3.3. ***Topology Theorem:***
For any metric space E, the following are open sets:

1. $\emptyset$
2. E
3. any union of open sets
4. any finite intersection of open sets

Corollary 3.1. *For any metric space E, the following are closed sets:*

1. $\emptyset$
2. E
3. any intersection of closed sets
4. any finite union of closed sets

So what? Why do we care about open and closed sets?

Let's prove some general facts:

Lemma 3.1. *The set* $(a, b) \subset \mathbb{R}$ *is* __________

Lemma 3.2. *The set* $[a, b] \subset \mathbb{R}$ *is* __________

Lemma 3.3. *The sets* $[a, \infty) \subset \mathbb{R}$ *(and* $(-\infty, a] \subset \mathbb{R}$*) are* __________

Lemma 3.4. *The sets* $(a, \infty) \subset \mathbb{R}$ *(and* $(-\infty, a) \subset \mathbb{R}$*) are* __________

Lemma 3.5. *The sets* $[a, b) \subset \mathbb{R}$ *(and* $(a, b] \subset \mathbb{R}$*) are* __________

Careful: The metric space matters! Some sets can be open in one metric space but not open in another!

Example 3.35. $(0, 1)$ *is open in* $\mathbb{R}$*. Is it open in* $\mathbb{R}^2$*?*

Lemma 3.6. $\mathbb{R} \subset \mathbb{R}^2$ *is* __________

Lemma 3.7. $\{(x, y) | x > 0, y > 0\} \subset \mathbb{R}^2$ *is* __________

3.3 Bounded Sets

In Real Analysis I, we talked about bounded sets. We said that a set was bounded if it was bounded above and bounded below. Unfortunately this language doesn't work well for general metric spaces. Intuitively what do you think this definition of boundedness would need to work in a general metric space.

Definition 3.11. *Let E be a metric space and let S be a subset of E. We define the* ***diameter*** *of S, denoted, $diam\,(S)$, in the following way*

- *If $S = \emptyset$, $diam\,(S) = 0$*
- *If $S \neq \emptyset$, $diam\,(S) = \infty$ if $\{d(a,b)|a,b \in S\}$ is not bounded above.*
- *If $S \neq \emptyset$, $diam\,(S) = \sup\{d(a,b)|a,b \in S\}$ if the supremum exists. In this case, we say S has* ***finite diameter.***

Example 3.36. *Let E be a metric space and let F be a finite subset of E. Prove F has finite diamter.*

Example 3.37. *Show that $(a,b) \subset \mathbb{R}$ has finite diameter.*

Example 3.38. *Show that $\mathbb{N} \subset \mathbb{R}$ has infinite diameter*

Theorem 3.4. *Let E be a metric space and $U \subseteq E$, then U has finite diameter if and only if $\exists x \in E$ and a $B(x,r)$ such that $U \subseteq B(x,r)$.*

Definition 3.12. *A set $U \subseteq E$ is* ***bounded*** *if either of the conditions from Theorem 3.4 hold. That is, either $\exists x \in E$ and a $B(x,r)$ such that $U \subseteq B(x,r)$ or U has finite diameter.*

Note for a whole metric space to be bounded, it needs to be bounded a subset of itself.

3.4 Convergent Sequences in Metric Spaces

Definition 3.13. *A sequence a_n ($\{a_n\}_{n=1}^{\infty}$)* ***converges*** *to $a \in E$ (written $\lim_{n\to\infty} a_n = a$) if and only if $\forall \epsilon > 0$, $\exists N > 0$ such that when $n > N$, $d(a_n, a) < \epsilon$. Such a sequence a_n is called a* ***convergent sequence.***

Does this definition makes sense from what we learned in Real Analysis I?

Examples:

Example 3.39. *Below are several statements involving sequences* ***in*** $\mathbb{R}$ *with a real number L. In each case, consider the statement as an alternate way to define $a_n \to L$. Think about the statements and provide an example of a sequence that satisfies the new definition, yet does not converge to L.*

1. The sequence a_n converges to L if $\forall \epsilon > 0$, $\exists n \in \mathbb{N}$ such that $d(a_n, L) < \epsilon$.

2. The sequence a_n converges to L if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that for some $n > N$, $d(a_n, L) < \epsilon$.

3. The sequence a_n converges to L if $\forall N \in \mathbb{N}$, $\exists \epsilon > 0$ so that for all $n > N$, $d(a_n, L) < \epsilon$.

4. The sequence a_n converges to L if $\forall N \in \mathbb{N}$ and $\forall \epsilon > 0$, $\exists n \in \mathbb{N}$ such that $d(a_n, L) < \epsilon$.

Theorem 3.5. *Let a_j be a sequence in metric space E, then the following statements are equivalent:*

1. *a_j converges to a.*
2. *For every open set U containing a, $\exists N \in \mathbb{N}$ such that for all $n > N$, $a_n \in U$.*
3. *$\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n > N$, then $a_n \in B_\epsilon(a)$*

Proposition 3.1. *Limits of Sequences are unique.*

Proof: Next HW

Example 3.40. *Prove or disprove the statement: Bounded Sequence are convergent.*

Example 3.41. *Prove or disprove the statement: Convergent Sequences are bounded.*

Proposition 3.2. *A sequence a_n in metric space E converges to a if and only if every subsequence of a_n also converges to a.*

We use the notation a_{n_k} to denote a subsequence of a_n

Pf. Omitted and very similar to the proof done in Real Analysis I. We use the fact that the index of the subsequence is greater than or equal to index of sequence. That is, for a sequence a_n, $n_k \geq k$ for all k.

Example 3.42. *Let's review of intuition about subsequences. content...*

1. [T/F] One subsequence of $(1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$ is $(1, 2, 5, 13, \dots)$.

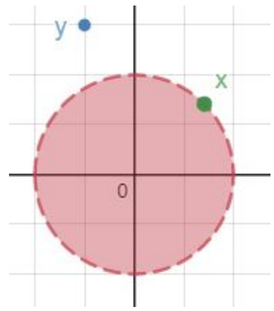
2. [T/F] One subsequence of $(1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$ is $(1, 2, 1, 5, 3, 13, 8, \dots)$.

3. [T/F] One subsequence of $(1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$ is $(1, 1, 2, 3, 5, 8, \dots)$.

4. [T/F] One subsequence of $(1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$ is $(1, 1, 2, 2, 5, 5, 13, 13, \dots)$.

3.5 Limit Points

Consider the picture of a set S to below. Our intuition tells us that the point x has a qualitatively different relationship to our set than the point y.



Notation: Given a set D and an element of the set D, say x_0,
the notation $D - \{x_0\}$ denotes the set:

Definition 3.14. *A sequence x_n is **distinct from** the point x_0 if $x_n \neq x_0$ for any n.*

Theorem 3.6. *Let $D \subseteq \mathbb{R}$ and $x \in D$. Then the following are equivalent:*

- $\exists x_n \in D - \{x\}$ *s.t.* $x_n \to x$.
- *There exists a sequence of distinct points of S converging to x.*
- *For all $r > 0$, $B(x, r)$ contains infinitely many points of D.*
- *For all $r > 0$, $B(x, r)$ contains a point of $D - \{x\}$*

Proof. ($1 \Rightarrow 2$) Let a_j be a sequence in $D - \{x\}$ which converges to x. For any sequence to converge, it either has a constant subsequence or subsequence of distinct terms. Because any constant subsequence of a sequence converging to x would be the sequence $(x, x, x, x, x, ...)$ and $a_j \in D - \{x\}$, so a_j cannot have a constant subsequence. Thus a_j has a subsequence of distinct terms which converges to x.
($2 \Rightarrow 3$) Suppose a_j is a sequence of distinct points of D that converges to x. Let $r > 0$, then $\exists N \in \mathbb{N}$ such that for all $n > N$, $a_n \in B(x, r)$. Since the terms of a_j are distinct for $j > N$, $B(x, r)$ contains infinitely many of points of D.
($3 \Rightarrow 4$) Since any non empty $B(x, r)$ about x contains infinitely many points in D, it also contains a point of D other than x. ($4 \Rightarrow 1$) For every $n \in \mathbb{N}$, pick $a_n \in B(x, \frac{1}{n}) \cap (D - \{x\})$. We will now show that $a_n \to x$. Let $\epsilon > 0$, let $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Fix $n > N$. Then $a_n \in B(x, \frac{1}{N})$. Thus $d(a_n, x) < \frac{1}{n} < \frac{1}{N} < \epsilon$. Thus $a_n \to x$. ∎

Definition 3.15. *A point $a \in E$ is a **limit point** of the set $A \subset E$ if and only if there exists a distinct sequence of points $p_n \subseteq A$ (wholly in A) such that $p_n \to a$.*

Corollary 3.2. *A point $a \in E$ is a **limit point** of the set $A \subset E$ if, for every $\epsilon > 0$, the ϵ-ball around a intersects A in a point other than a; that is, if $(A \cap B_\epsilon(a)) - \{a\} \neq \emptyset$.*

Example 3.43. *The endpoints of the open interval* $(0, 1)$ *are limit points.*

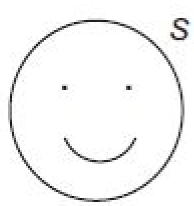
Example 3.44. *Every real number is a limit point of* $\mathbb{Q}$*.*

Example 3.45. *What are the limit points of the finite set* $F = \{1, 2, 7, 9, 279\}$*?*

The distinction between D and $D - \{x_0\}$ is very important. The following examples explore this

Example 3.46. *Consider a set* $D \subseteq \mathbb{R}$*. Show that every* $x \in D$ *has the property that there is a sequence of points in* D *converging to it.*

Example 3.47. *Consider the "Smiley face" below. In this figure the subset $D \subseteq \mathbb{R}^2$ consists of the boundary of the circle, the "smile," and the two "eyes."*



a)Which points in the smiley set satisfy the definition of limit point?

b) Which don't?

c) Give your intuition about why the distinctions make sense.

Example 3.48. *For the next problems, prove or provide a counterexample.*

1. [T/F] If a is a limit point of A, then $a \in A$.

2. [T/F] If a is a limit point of A, then there is a sequence $(a_n) \subset A - \{a\}$ with $a_n \to a$.

3. [T/F] If there is a sequence $(a_n) \subset A - \{a\}$ with $a_n \to a$, then a is a limit point of A.

4. [T/F] If the set A is closed, then it contains all its limit points.

5. [T/F] If the set B is open, then it contains all its limit points.

6. [T/F] If the set C contains all its limit points, then it is closed.

7. [T/F] If the set D contains all its limit points, then it is open.

8. [T/F] If the set E contains all its limit points, then it is not open.

Example 3.49. *Prove that if* $\lim_{n\to\infty} a_n = L$ *then the set* $\{L, a_1, a_2, a_3, ...\}$ *is closed.*

Lemma 3.8. *A set $O \subset M$ is open if and only if none of its points are limits of its complement.*

Proof: Homework!

Definition 3.16. *Given an metric space, E, we say a set $D \subseteq E$ is **dense** if every point of E is either an element of D or a limit point of D.*

Example 3.50. $\mathbb{Q}$ *and* $\mathbb{Q}^c$ *are dense subsets of* $\mathbb{R}$. *What is a countable, dense subset of* $\mathbb{R}^2$*?*

3.6 Isolated Points

Definition 3.17. *A point $x \in D$ is said to be an **isolated point** if there exists an open ball centered at x such that x is the only point of D in the ball.*

Draw the Picture:

Proposition 3.3. *A point $x_0 \in D$ is either an isolated point or a limit of point of D.*

Example 3.51. *HW: Let* $S = \{(\frac{(-1)^n \cdot n}{n+1}) | n = 1, 2, 3, ...\}$
a) Find the limit points of S.

b) Is S a closed set? (prove or disprove)

c) Is S an open set? (prove or disprove)

d) Does S contain any isolated points?

3.7 Closure

Definition 3.18. *Let E be a metric space and let $S \subseteq E$. The intersection of all closed subsets of E which contain S is called the **closure of S**. We denote this set by $\bar{S}$.*

Proposition 3.4. *$\bar{S}$ is a subset of every closed set that contains S.*

Proposition 3.5. *$\bar{S}$ is a closed set.*

Proposition 3.6. *S is closed if and only if $S = \bar{S}$.*

Example 3.52. *Let $x \in \bar{X}$. Prove the following statements are equivalent:*

1. There exists a sequence in S that converges to x.
2. Every open ball around x contains a point of S.
3. For every open set U containing x, $U \cap S \neq 0$.
4. $x \in \bar{S}$

Proof: Homework

Example 3.53. *Prove that $\bar{S}$ is the union of S and all limit points of S.*

Proof: Homework

Example 3.54. *Prove if $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.*

Example 3.55. *Prove $\overline{A \cup B} = \bar{A} \cup \bar{B}$.*

Example 3.56. *Prove $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$. Give an example where $\bar{A} \cap \bar{B}$ is not necessarily a subset of $\overline{A \cap B}$.*

3.8 Interior

Definition 3.19. *Let E be a metric space and let $S \subseteq E$. A point $x \in S$ is called an **interior point** of S provided that there is an open ball around x that is totally contained within S. The set of all interior points of S is called the **interior of S**. We denote this set by S° or $Int(S)$.*

Draw the picture:

Proposition 3.7. *Let E be a metric space and let $S \subseteq E$. Then S° is an open subset of E.*

Proof. Let p be an interior point of S. Then $\exists r > 0$ s.t. $B(p,r) \subseteq S$. Let $z \in B(p,r)$. Choose s such that $B(z,s) \subseteq B(p,r) \subseteq S$. Since this ball is entirely contained in S, z is also an interior point of S. Thus $B(p,r) \subseteq S^\circ$. ∎

Proposition 3.8. *Let E be a metric space and let $S \subseteq E$. Then every open subset of E that is contained in S is contained in S°.*

Proof. Suppose U is an open subset of E such that $U \subseteq S$. Then for each $x \in U$, we can choose $r > 0$ such that $B(x,r) \subseteq U \subseteq S$. Thus x is an interior point of S and $U \subseteq S^\circ$. ∎

Proposition 3.9. *Let E be a metric space and let $S \subseteq E$. Then the union of all open subsets of E that are contained in S is equal to S°.*

Proof. Since every open subset of E that is contained in S is contained in S°, it follows that the union of all such sets must also be contained in S°. On the other hand in order to show that S° is contained in the union of all open sets, we need only to note that any interior point of S is, trivially, in S and therefore that $S^\circ \subseteq S$. Since S° is an open subset of E and is contained in S, it is clearly a subset of the union of all such sets. ∎

Proposition 3.10. *Let E be a metric space and let $S \subseteq E$. Then S is open if and only if $S = S^\circ$.*

Proof. $\Rightarrow$: If S is open, it is one of the open subsets of E that is contained in S. Thus (by part (2)) $S \subseteq S^\circ$. As we have already noted in 3, $S^\circ \subseteq S$.

$\Leftarrow$: If $S^\circ = S$, then S is open in E because the interior is always open.

∎

Proposition 3.11. *Let E be a metric space and let $S \subseteq E$. Then $S^\circ = (\overline{S^c})^c$.*

Proof. To show $S^\circ \subseteq (\overline{S^c})^c$: Let $x \in S^\circ$ By definition, this means that there is an open ball $B(x, r) \subseteq S$. Thus $x \notin S^c$ and x is not a limit point of S^c which means $x \notin \overline{S^c}$. Therefore $x \in (\overline{S^c})^c$.

To show $(\overline{S^c})^c \subseteq S^\circ$: Suppose $x \in (\overline{S^c})^c$. This means $x \notin \overline{S^c}$ which means $x \notin S^c$ and thus x is not a limit point of S^c. Therefore $\exists r > 0$ s.t. $B(x, r) \subseteq S$ which means $x \in S^\circ$. ■

Example 3.57. *Is it always true that Int(S) =Int$(\overline{S^c})$?*

Definition 3.20. *Let E be a metric space and let $S \subseteq E$. We define the* ***boundary of S*** *to be $\overline{S} \cap \overline{(S)^c}$. We denote it as ∂S.*

Draw the picture:

Example 3.58. *[T/F] ∂S is a closed set.*

Example 3.59. *[T/F] ∂S is an open set.*

Example 3.60. *[T/F]* $x \in \partial S$ *if and only if* $\forall \epsilon > 0$, $B_\epsilon(x) \cap S \neq \emptyset$ *and* $B_\epsilon(x) \cap S^c \neq \emptyset$

Example 3.61. *[T/F]* S *is closed if* $\partial S \subseteq S$.

Example 3.62. *[T/F] If* $\partial S \subseteq S$, S *is closed.*

Example 3.63. *[T/F] If* S *is open then* $\partial S \cap S = \emptyset$.

Example 3.64. *[T/F] If $\partial S \cap S = \emptyset$, then S is open.*

Example 3.65. *[T/F] A metric space is the disjoint union of ∂S, $\overline{S}$, and S°.*

4 Complete Metric Spaces

Here are some examples to keep in mind during this section.

- $\mathbb{R}$ is nice because it "has no gaps."
- $\mathbb{Q}$ is not so nice because it *does* "have gaps."
- $(0, 1]$ is not so nice because it doesn't contain all of its limit points. That is sequences like $a_n = \frac{1}{n}$ can "leak out" in the limit.

In this section we are going to talk about a property of metric spaces called **completeness.** In Real Analysis I, we talked about how the Completeness Axiom (which states any non-empty, bounded above set in $\mathbb{R}$ has a least upper bound) describes a difference between $\mathbb{Q}$ and $\mathbb{R}$ since we can have bounded above sets in $\mathbb{Q}$ that don't have sups. We call this axiom the "Completeness Axiom" because it guarantees that $\mathbb{R}$ has a metric space condition called "completeness."

Basic Idea: We want to talk about whether or not there exists "pinprick-holes" in a metric space. Some metric spaces aren't ordered, so nothing parallel to the Least Upper Bound Axiom is possible. Thus we need a new approach to describe this property.

It turns out our old friend, the Cauchy Sequence, can help. Let's recall our definition of a Cauchy sequence.

Definition 4.1. *A sequence a_n in a metric space (E, d) is called **Cauchy** if and only if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. for $n, m > N$, then $d(a_n, a_m) < \epsilon$*

Draw the intuitive picture for this definition:

Proposition 4.1. *Every convergent sequence is a Cauchy Sequence.*

Proof. Let a_n be a convergent sequence with limit, a and let $\epsilon > 0$. Because a_n converges, there exists an $N \in \mathbb{N}$ s.t. if $n > N$, $d(a_n, a) < \dfrac{\epsilon}{2}$. Then for $n, m > N$,
$d(a_n, a_m) \leq d(a_n, a) + d(a, a_m) < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$. ∎

Caution: In $\mathbb{R}$, if we have a cauchy sequence, it is also __________, but this isn't true in every metric space.

Example 4.1. *Give a few examples of Cauchy sequences that are not convergent to a point in a given metric space.*

Definition 4.2. *A metric space, E, is called* ***complete*** *if and only if every Cauchy sequence in E converges to an element in E.*

Example 4.2. *What are some examples of Complete Metric Spaces?*

Lemma 4.1. *Any subsequence of a Cauchy sequence is Cauchy.*

Proof. Let a_n be Cauchy. Let $\epsilon > 0$. Let a_{n_k} be a subsequence of a_n. Choose $N \in \mathbb{N}$ s.t. if $n > N$, $d(a_n, a) < \epsilon$. Fix $n, m > N$. Then $j, m \geq m > N$ and $n_j \geq j > N$. Therefore $d(a_{n_j}, a_{n_m}) < \epsilon$. ■

Proposition 4.2. *Every Cauchy sequence is bounded.*

Proposition 4.3. *If a_n is a Cauchy sequence which has a convergent subsequence, then a_n must converge.*

Proposition 4.4. *Every closed subspace of a complete sequence is complete.*

Proof: Homework.

5 Compact Metric Spaces

You know is really nice? When we have finite sets. With finite sets we have a lot of nice properties like:

1. Every real-valued function from a finite domain has a maximum or minimum value.
2. Given a finite set, F, there exists points $x, y \in F$ s.t. $d(x, y) =$diam(F)
3. Given a finite subset, F, of a metric space E and a point in E but not in F, $\exists y \in F$ s.t. $d(x, y) \leq d(x, f)\ \forall f \in F$. (There exists an element of F that is closest to x)
4. Every sequence that has a finite range has a convergent subsequence.

Example 5.1. *For each of the following properties, give an example which proves that these properties do not hold for any general set (non-finite sets).*

You know what else is really nice? Closed and bounded intervals? Remember, there are a lot of nice theorems and properties of functions which we learned about in Calculus I and then proved in Real Analysis I.

Our goal for this section is to discuss a property of metric spaces which shares a lot of the nice mathematical properties that finite sets have and is a generalization of the "closed and bounded" properties of $\mathbb{R}$. This property is called "compactness." Unfortunately it is also perhaps the most abstract definition we have had to deal with so far. There are also a number of equivalent ways to describe this property! So hang in there!

Definition 5.1. *Let E be a metric space and let $S \subseteq E$. If $U = \{U_i\}_{i \in I}$ is in a collection of open subsets of E and $S \subseteq \bigcup_{i \in I} U_i$, then we say U is an* ***open cover*** *for S. Any sub-collection of U whose union still contains S is called a* ***sub-cover*** *for S.*

Example 5.2. *Examples of Open Covers*

1. Let E be a metric space and $S \subseteq E$. Let r be a fixed positive real number. then the collection of open balls of radius r around each point in S, $U = \{B_r(x)\}_{x \in S}$ is an open cover of S.

2. Let E be a metric space and $S \subseteq E$. Let $a \in E$. Then $U = \{B_n(a)\}_{n=1}^{\infty}$ is an open cover of S. (It also covers all of E!)

3. Consider the metric space $\mathbb{R}$ and the $S = (0, 1) \subseteq \mathbb{R}$. Let $r_1, r_2, r_3, ...$ be an enumeration of rational numbers in $(0, 1)$ (remember $\mathbb{Q}$ is countable!). For each $n \in \mathbb{N}$, let $U_n = (r_n - \frac{1}{4}, r_n + \frac{1}{4})$. Then $U = \{U_n\}_{n=1}^{\infty}$ is an open cover of S.

   Notice that the sub-collection $\{\frac{1}{4} - \frac{1}{4}, \frac{1}{4} + \frac{1}{4}), (\frac{1}{2} - \frac{1}{4}, \frac{1}{2} + \frac{1}{4}), (\frac{3}{4} - \frac{1}{4}, \frac{3}{4} + \frac{1}{4})\} = \{(0, \frac{1}{2}), (\frac{1}{4}, \frac{3}{4}), (\frac{1}{2}, 1)\}$ is a sub-cover U which covers all of S.

Definition 5.2. *Let E be a metric space we say a subset S of E is **compact** if <u>every</u> open cover of S has a <u>$finite$</u> sub-cover.*

In other words, if $U = \{U_i\}_{i \in I}$ and $S =\subseteq \bigcup_{i \in I} U_i$, then $\exists U_{i_1}, U_{i_2}, U_{i_3}, ..., U_{i_n} \in U$ such that

$$S \subseteq \bigcup_{j=1}^{n} U_{i_j}$$

Definition 5.3. *A metric space is said to be a **compact metric space** if it is compact as a subset of itself.*

Draw the Picture:

Question: Does Example 5.2 #3 tell us that $(0, 1)$ is compact? Why or why not?

Note: This definition of compactness is, arguably, more easily used to show that a space S is *NOT* compact since all you need to do is concoct __________ open cover which has no finite sub-cover.

Example 5.3. *Basic Examples and Non-examples of Compact Sets*

a) $[T/F]$ $\mathbb{R}$ is compact.

b) $[T/F]$ $(0,1)$ is compact.

c) $[T/F]$ Given a sequence of distinct terms in metric space E, $a_1, a_2, a_3, ...$, which converge to a, $S = \{a, a_1, a_2, a_3, ...\}$ is compact.

d) $[T/F]$ $\mathbb{R}^2$ is compact.

e) $[T/F]$ $[0,1]$ is compact.

f) $[T/F]$ Finite sets are compact.

g) $[T/F]$ $[0, 1)$ is compact.

Theorem 5.1. *The closed interval* $[a, b]$ *is compact.*

Proposition 5.1. *Any closed subset of a compact metric space is compact.*

Proposition 5.2. *A compact subset of a metric space is bounded. In particular a compact metric space is bounded.*

Proposition 5.3. *The Nested Interval Property (NIP): For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n]$. Assume also that each I_n contains I_{n+1}. Then the intersection of these nested, closed intervals is non-empty; that is $\cap_{n=1}^{\infty} I_n \neq \emptyset$.*

We have a similar one for the general case:

Theorem 5.2. *Let X be a compact metric space then the intersection of these nested, closed subsets is non-empty. That is if $\{C_i\}$ is a nested sequence of non-empty closed subsets of X (That is $C_1 \supseteq C_2 \supseteq C_3 \supseteq ...$) then $\cap_{n=1}^{\infty} C_n \neq \emptyset$.*

Proof. This can be proved using the contrapositive.

5.1 Cluster Points

Definition 5.4. *Let $S \subseteq E$ we say a point $p \in E$ is a* ***cluster point of S*** *if and only if every open ball centered at p contains infinitely many points of S.*

How are limit points connected to cluster points?

Note: A cluster point of S may be inside or outside of S.

Example 5.4. *For $S = (0, 1]$. What are the cluster points?*

Example 5.5. *For $S = (0, 1] \cup \{2\}$. What are the cluster points?*

Example 5.6. *For a metric space E with the discrete metric what are the cluster points?*

Theorem 5.3. *Any infinite subset of a compact metric space has at least one cluster point.*

Corollary 5.1. *Any sequence of points in a compact metric space has a convergent subsequence.*

Corollary 5.2. *A compact metric space is complete.*

Corollary 5.3. *A compact metric space is closed.*

Example 5.7. *Let E be a metric space and suppose F is a closed subset of E and K is a compact subset of E. Prove $K \cap F$ is compact.*

Example 5.8. *$[T/F]$ Any arbitrary intersection of compact sets is compact.*

Example 5.9. $[T/F]$ *A union of a finite number of compact sets is compact.*

Theorem 5.4. *Let E be a metric space and $S \subseteq E$, then the following are equivalent:*

1. S is compact.
2. Every sequence in S has a subsequence that converges to a point in S.
3. Every infinite subset of S has a limit point in S.

Theorem 5.5. ***Heine-Borel Theorem:*** *A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded. (This is also true for $\mathbb{R}^n$, but we will not prove this.)*

Homework Questions:

Definition 5.5. *We define the metric space ℓ_∞ (called "little-ell-infinity") to be the set of all bounded sequences of real numbers with the following metric:*
Given $\boldsymbol{x} = (x_1, x_2, x_3 ...)$ and $\boldsymbol{y} = (y_1, y_2, y_3, ...)$, $d(\boldsymbol{x}, \boldsymbol{y}) = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$

a) Use this metric to compute the distances between the following pairs of sequences:
i) $(1, 0, 0, 0, 0, ...)$ and $(0, 1, 0, 0, 0, 0, ...)$

ii) $(1, 0, 1, 0, 1, 0, 1, ...)$ and $(0, -1, 0, -1, 0, -1, ...)$

iii) $(1, 2, 3, 1, 2, 3, ...)$ and $(-1, -2, -3, -1, -2, -3, ...)$

iv) $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...)$ and $(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...)$

b) Prove that the closed ball of radius 1 about $\mathbf{0} = (0, 0, 0, 0, ...)$ is not compact. To do so, consider the sequence (of sequences) $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, ...\}$ in ℓ_∞ where $\mathbf{e_i} = (e_{i1}, e_{i2}, e_{i3}, ...)$ is given by

$$e_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Show that $(\mathbf{e_i})$ is, indeed, in $B_1(\mathbf{0})$ and that $B_1(\mathbf{0})$ is 1-separated which means that $d(x, y) \geq 1$ for all x and y such that $x \neq y$. This implies that this closed ball is not compact.

Example 5.10. *For a general metric space E and subspace S, which of the following are true and which are false?*

a) If S is compact then S is closed.

b) If S is closed then S is compact.

c) If S is compact, then S is complete.

d) If S is complete then S is compact.

e) If S is closed and bounded then S is compact.

Example 5.11. *Consider the metric space $\mathbb{R}^2$, but consider the metric given by*

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2|, & x_1 = x_2 \\ 1 + |y_1 - y_2 & x_1 \neq x_2 \end{cases}$$

a) Give a description of all possible types of open balls in this setting. [Hint: Consider the cases of $r \leq 1$ and $r > 1$ separately.]

b) Is $\mathbb{R}^2$ with this metric complete?

c) Is our space compact?

6 Connected Metric Spaces

Let's recall the Intermediate Value Theorem from Real Analysis I and Calculus I:

Theorem 6.1. ***Intermediate Value Theorem:*** *Suppose f is continuous on $[a,b]$ (a closed and bounded domain), then for any y s.t. $f(a) < y < f(b)$, $\exists \hat{x} \in (a,b)$ s.t __________.*

Draw the picture:

Why is this theorem so useful?

1.

2.

While we have proved this theorem in Real Analysis I, we wish to explore the fundamental mathematical principles which are under the surface of this theorem.

- $[a,b]$ is a *connected set* of $\mathbb{R}$.
- a continuous image of a connected set is connected. Thus $f([a,b])$ is connected.

Intuitively, draw a picture of a disconnected set vs a connected set.

What about $\mathbb{R}$?

Let's think about disconnected sets. If C is a disconnected set made up of two disjoint pieces A and B, what must we need?

So if C is made up of disjoint A and B, so $C = A \cup B$ (a disjoint union can be denoted by $A \sqcup B$), we must have that B contains no limit point of A and A contains no limit point of B. Before we formally define connected and disconnected, let's first introduce a mind dizzy-ing definition.

First note if we have is a metric space, E, and $S \subseteq E$. Then S inherits a metric from E and can be thought of as a metric space in and of itself.

Definition 6.1. *Suppose E is a metric space and $S \subseteq E$. (So S inherits a metric from E is a metric space.) We say $U \subseteq S$ is* ***relatively open in S*** *if and only if $\exists W \subseteq E$ such that W is open in E and $U = W \cap S$.*

Careful: Note that U may not be open in E.

Example 6.1. *Let $E = \mathbb{R}$ and $S = [0, 1]$.*
a) [T/F] $A = (\frac{1}{2}, 1]$ is an open subset of $\mathbb{R}$.

b) [T/F] $A = (\frac{1}{2}, 1]$ is an open subset of S.

Definition 6.2. *Suppose E is a metric space and $S \subseteq E$. (So S inherits a metric from E is a metric space.) We say $C \subseteq S$ is* ***relatively closed in S*** *if and only if $\exists K \subseteq E$ such that K is closed in E and $C = K \cap S$.*

Careful: Note that C may not be closed in E.

Example 6.2. *Let $E = \mathbb{R}$ and $S = (0, 2]$.*
a) [T/F] $A = (0, 1]$ is a closed subset of $\mathbb{R}$.

b) [T/F] $A = (0, 1]$ is an closed subset of S.

Definition 6.3. *Sets that are both open and closed are called **clopen** sets.*

Definition 6.4. *Let E be a metric space and $S \subseteq E$. We say two sets A and B **separate** S if they satisfy the following conditions:*

1. $A \cap B \cap S = \emptyset$
2. $S \cap A \neq \emptyset$
3. $S \cap B \neq \emptyset$
4. $S \subset A \cup B$

Draw the Picture:

Definition 6.5. *A metric space E is said to be **disconnected** if $\exists$ non-empty clopen subsets, $A, B \subseteq E$ such that $E = A \sqcup B$ (That is, E is the disjoint union of A and B).*

Note: Another way to say this is to say A and B *separate* E.

Definition 6.6. *A metric space E is said to be **connected** if it is not disconnected.*

Note: That is, we cannot find two sets which separate E.

Example 6.3. *Which of the following sets are connected:*
a) $[1, 2] \cup [3, 4]$

b) $[0, 2]$

c) $(0, 2)$

Theorem 6.2. *Let E be a metric space. Then the following are equivalent:*

1. E is connected.
2. The only subsets of E that are clopen are E and $\emptyset$.
3. There does NOT exist nonempty, disjoint open sets A and B such that $E = A \sqcup B$.
4. There does NOT exist nonempty, disjoint closed sets A and B such that $E = A \sqcup B$.

Proof:

Example 6.4. *For the following sets identify which of the sets are open, which are closed, clopen, or neither open nor closed. Also determine which are connected.*
a) $E = \{(x, y) : y \neq 0\}$

b) $E = \{(x, y) : x^2 + 4y^2 \leq 1\}$

c) $E = \{(x, y) : y \geq x^2, 0 \leq y < 1\}$

d) $E = \{(x, y) : x^2 - y^2 > 1, -1 < y < 1\}$

Example 6.5. *Is* $\mathbb{Z} \subseteq \mathbb{R}$ *connected?*

Proposition 6.1. *Let A be a connected subset of E. Show that $\bar{A}$ is connected (the closure of A).*

Example 6.6. *Consider the metric space $\mathbb{R}^2$, but use the metric given by*

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |y_1 - y_2|, & x_1 = x_2 \\ 1 + |y_1 - y_2 & x_1 \neq x_2 \end{cases}$$

Last section we discussed some properties of this metric space. Is this metric space connected?

7 Continuous Functions

We can generalize our definitions from Real Analysis I for continuous functions on general metric spaces.

Definition 7.1. ***Pointwise Continuity:*** *Let $f : E \to \tilde{E}$ where E and $\tilde{E}$ are metric spaces with metrics d and $\tilde{d}$ respectively. And let $a \in E$. We say that f* ***is continuous at*** *a if $\forall \epsilon > 0$ there exists a $\delta > 0$ such that whenever $d(p, a) < \delta$ (and $x \in E$), we have $\tilde{d}(f(p), f(a)) < \epsilon$.*

Definition 7.2. ***Global Continuity:*** *We say f is a continuous function on E iff f is continuous at every point $a \in E$.*

In other words, $\lim\limits_{x \to a} f(x) = f(a)$.

We can also classify continuity in terms of sequences:

Theorem 7.1. *Continuity Theorem Given a function $f : D \to \mathbb{R}$ and a limit point a of D. TFAE (The following are equivalent):*

- *f is continuous at a. That is $\lim\limits_{x \to a} f(x) = f(a)$*
- *For every sequence $(x_n) \subset A$ s.t. $(x_n) \to a$, it follows that $(f(x_n)) \to f(a)$.*

But wait! There is more. I bet we can classify continuity in terms of balls too!

Definition 7.3. *Given $f : E \to \tilde{E}$, f* ***is continuous at*** *$a \in E$ iff for every open ball $B_\epsilon(f(a))$, $\exists B_\delta(a)$ such that $f(B_\delta(a)) \subseteq B_\epsilon(f(a))$*

i.e. $(B_\delta(a) \subseteq f^{-1}(B_\epsilon(f(a)))$
Draw the Picture:

Global Continuity Topology Definition:

Theorem 7.2. *f is a continuous function iff every open subset, U, of $\tilde{E}$, the inverse image $f^{-1}(U) = \{a \in E : f(a) \in U\}$ is open.*

Draw the Picture:

Proof: $\Leftarrow$ Suppose $\forall U \subseteq \tilde{E}$, $f^{-1}(U)$ is open in E. This means for all $\epsilon > 0$, $f^{-1}(B_\epsilon(f(p_0))$ is an open subset of E which contains p_0. This means there exists a $\delta > 0$ such that $B_\delta(p_0) \subseteq f^{-1}(B_\epsilon f(p_0)))$. So if $p \in E$ and $d(p, p_0) < \delta$, then $\tilde{d}(f(p), f(p_0)) < \epsilon$. Thus f is continuous.

$\Rightarrow$

Corollary 7.1. *Let $f : E \to \tilde{E}$, $f$is a continuous function iff if every closed subset, V, of $\tilde{E}$, the inverse image $f^{-1}(V) = \{a \in E : f(a) \in V\}$ is closed.*

Proof. Follows from previous theorem.

Question 1: Let $f : E \to \tilde{E}$ be continuous and let $U \subseteq E$ be open. Must $f(U)$ be open?

Question 2: Let $f : E \to \tilde{E}$ be continuous and let $V \subseteq E$ be open. Must $f(V)$ be closed?

Example 7.1. *Let $f : E \to \tilde{E}$ be a constant function. Is f continuous? Prove or disprove.*

Example 7.2. *Let $f : E \to E$ be the identity function $f(a) = a$. Is f continuous? Prove or disprove.*

Example 7.3. *Let* $f : \mathbb{R} \to \mathbb{R}$. *where* $f(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \notin \mathbb{Q} \end{cases}$ *Where is* f *continuous? Explain.*

Proposition 7.1. *Let* $f : E \to E'$ *and* $g : E' \to E''$ *both be continuous, then* $g \circ f$ *is continuous.*

Draw the Picture:

Proof. Let E, E', E'' all be metric spaces with respective metrics d, d', d'' denote the three metrics. Let $p_0 \in E$ and let $\epsilon > 0$. Since g is continuous, g is continuous at $f(p_0)$, then $\exists \delta > 0$ such that if $q \in E'$ and $d'(q, f(p_0)) < \delta$, then $d''(g(q), g(f(p_0)) < \epsilon$. Since f is also continuous at p_0, $\exists \eta > 0$ such that if $p \in E$ and $d(p, p_0) < \eta$, then $d'(f(p), f(p_0)) < \delta$. Thus if $p \in E$ and $d(p, p_0) < \eta$, then $d''(g(f(p)), g(f(p_0))) < \epsilon$. Therefore $g \circ f$ is continuous at p_0. ■

7.1 Properties of Continuous Functions

Theorem 7.3. *Let $f : E \to \tilde{E}$ be continuous and let $K \subseteq E$ be a compact subset of E, then $f(K)$ is also compact.*

Proof. Let $f : E \to \tilde{E}$ be continuous and let $K \subseteq E$ be a compact subset of E. We want to show that if $f(K) = \{f(p)|p \in K\}$ has an arbitrary open cover, it has a finite subcover. Let $\{O_i\}_{i\in I} \subseteq \tilde{E}$ be such an open cover of $f(K)$. Since f is continuous, for each $i \in I$, $f^{-1}(O_i)$ is open in K. So for any $p \in K$, $f(p) \in O_i$ for some $i \in I$ and thus $p \in f^{-1}(O_i)$. Thus $K \subseteq \bigcup_{i\in I} f^{-1}(O_i)$. Since K s compact, this open cover has a finite subcover. Thus $K \subseteq \bigcup_{i=1}^{n} f^{-1}(O_i)$. Thus $f(K) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(O_i)) \subset \bigcup_{i=1}^{n}(O_i)$. Therefore $f(K)$ is compact. ■

Theorem 7.4. Extreme Value Theorem Revisited: *Let $f : E \to \tilde{E}$ be continuous and let $K \subseteq E$ be a nonempty compact subset of E, then f is bounded.*

In other words, if $f : E \to \mathbb{R}$ is continuous and $K \subseteq E$ is a nonempty compact subset of E, $\exists M, m \in K$ such that $\forall z \in K$, $f(m) \leq f(z) \leq f(M)$.

Proof. Any compact subset of a metric space is bounded. Thus $f(K)$ is bounded. ■

Theorem 7.5. *Let $f : E \to \tilde{E}$ be continuous and let E be a compact metric space, then f is uniformly continuous.*

Proof: Omitted.

Theorem 7.6. *Let $f : E \to \tilde{E}$ be continuous and let E be a connected metric space, then $f(E)$ is also connected.*

Note the IVT is a corollary to this theorem.
Proof:

Example 7.4. *What subspaces in* $\mathbb{R}$ *are connected?*

Example 7.5. *Let* C *be a subset of* $\mathbb{R}^2$ *defined as* $([0,1] \times \{0\}) \bigcup (\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\} \times [0,1])$. *Sketch* C *and you will see what it is called "The Comb." Is* C *connected?*

Example 7.6. *Let* D *be a susbset of* $\mathbb{R}^2$ *defined as* $D = C \ (\{0\} \times (0,1))$ *Sketch* D *(the "Deleted Comb"). Is* D *connected.*

7.2 Path Connected

Definition 7.4. *Let E be a metric space. We say E is **path-connected** (or arcwise connected or pathwise connected) iff for any two points $p, q \in E$, $\exists$ a continuous function $f : [0, 1] \to E$ with $f(0) = p$ and $f(1) = q$.*

Idea: Think of tracing a continuous curve from any point p to any point q.

Example 7.7. *What subspaces in $\mathbb{R}$ are path-connected?*

Example 7.8. *Is C, The Comb, path connected?*

Example 7.9. *Is D, the The Deleted Comb, path-connected?*

Theorem 7.7. *Let $f : E \to \tilde{E}$ be continuous. If E is path-connected, then $F(E)$ is path connected.*

Example 7.10. *[T/F] If E is path connected, E is connected.*

Example 7.11. *[T/F] If E is connected, E is path-connected.*